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MULTIPLE EQUILIBRIA AND STRUCTURAL TRANSITION OF NON-MONOCENTRIC URBAN CONFIGURATIONS*

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A model of non-monocentric urban land use is presented, which requires neither employment nor residential location to be specified *a priori*. It is shown that the model is capable of yielding multicentric pattern as well as monocentric and dispersed patterns, and that the model generally yields multiple equilibria under each fixed set of parameter values. It is also shown that the city may undergo a catastrophic structural transition when the parameters take critical values.

1. Introduction

In the development of the economic theory of urban land use, a pivotal event was the introduction of the concept of *bid-rent curves* by Alonso in the early 1960's. Alonso (1964) defines bid-rent for a household (firm) as functions of the distance from the city center and the utility (profit) level of the household (firm). But a question arises: in our attempt to obtain the equilibrium locations of all the households and firms in a city, how should we define the *city center*? That is, when we do not know the location of any household or firm, how can we determine *a priori* the location of the city center? One way to get around this problem is to introduce the assumption of *monocentricity*. By monocentricity we mean that the city under study is assumed to have a single, prespecified center of production activities, the CBD, which has a fixed size and employs the city's entire labor force. This assumption of monocentricity can greatly simplify the study of urban land use, and has been adopted in most of the works emanating from the so-called school of the 'New Urban Economics'.

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However, from both the viewpoints of theory and reality, this assumption of monocentricity has drawbacks. First, from the viewpoint of theoretical completeness, the centrality or non-centrality of a city should be explained within the framework of the model. A more satisfactory model would yield a spatial structure of the city in which the locations of households and firms are endogenously determined, without assuming the location of either *a priori*.

Secondly, from the viewpoint of reality, a monocentric urban land use pattern seems to be untenable. Many studies [e.g., Kemper and Schmenner (1974) and Mills (1972)] have shown the pervasive tendency of increasing decentralization of both households and firms and the consequent decline of the role of the CBD as a single focus of employment. Furthermore, Odland (1978) conducted a statistical test of the hypothesis of monocentricity and concluded that this assumption may not be sustainable. Thus, it can be claimed that the concept of a monocentric city is not a satisfactory description of certain modern cities.

Consequently, the development of non-monocentric models of urban land use is needed both from the viewpoints of theoretical completeness and practical usefulness. At present, there exist several pioneering efforts to develop *non-monocentric models* of urban land use which do not assume an *a priori* location of either employment or households [Beckmann (1976), Borukhov and Hochman (1977), Capozza (1976), Odland (1976, 1978), Ogawa and Fujita (1978, 1979) and Ratford (1973)].[1] Unfortunately, these attempts are still far from the goal of constructing a general model of non-monocentric urban land use.[2]

It is the purpose of this paper to present a model of non-monocentric urban land use within the framework of static microeconomic theory. The model proposed here requires neither employment nor residential location to be specified *a priori*, and yields various different types of urban spatial structure, including monocentric and multicentric patterns, depending on the values of the model parameters. Hence, the present study constitutes a contribution to the development of a general theory of non-monocentric urban land use. However, it must be noted that the model presented here is essentially experimental in nature: it is purely static and one-dimensional and

[1]We here exclude those models (the so-called *multicentric models*) which prespecify the number and locations of employment centers in the city.

[2]Pioneering models by Beckmann (1976) and Borukhov and Hochman (1977) contain only one sector. Models by Capozza (1976) and Ogawa-Fujita (1978, 1979) are limited in generality since they cannot generate multicentric city patterns at the equilibrium. The models formulated by Odland (1976, 1978) are more general, but unfortunately the analysis is not fully developed. The model by Amson (1976) is based on social physics, and assumes a fixed center. The main difference between the model in Ogawa-Fujita (1978, 1979) and the model in this paper is that, while agglomeration economies in the former model are represented by savings in transport costs, those in the latter model are based on external economies among business firms which are conceptually more abstract but more general.

is based on a number of strong simplifying assumptions. For this reason, further investigation and analysis of the ideas presented in this paper will have to be carried out before our model could be applied to the problem of land use in real cities.

In the next section, we present a non-monocentric model of urban land use which focuses on the economic interactions among households and business firms in the city. The spatial configuration of the city is treated as the outcome of these interactions between business firms, which favor concentration by reason of agglomeration economies, and households, which follow closely the employment distribution (because of the costs of commuting from residences to job sites), with the consumption of urban land as the mediator of the balance. To represent agglomeration economies among firms, we introduce the concept of the *locational potential*. It varies among locations depending on the degree of concentration or dispersion of business firms, and hence, serves as an index of locational advantage for production due to agglomeration economies. Using this concept together with the generalized notion of bid land rent, we formulate an equilibrium model of non-monocentric urban land use.

Each equilibrium solution of the model is characterized by household distribution, business firm distribution, land rent profile, wage profile, commuting pattern, utility level of household, all of which are determined simultaneously, under a given set of exogenous parameters. However, in spite of the simplicity of this model, qualitative solutions for the problem are not readily obtainable. To get around this difficulty, in section 3, we first analytically derive the conditions for the existence of monocentric, non-monocentric and multicentric urban configurations, respectively. Then, we conduct a numerical analysis to determine the range of parameter space in which each specific type of urban spatial configuration occurs. The major parameters here are commuting rate for the households, production level and locational potential parameters for the business firms. It is shown that the model is capable of yielding multicentric patterns as well as monocentric and non-centric patterns, and that the model generally yields multiple equilibria under each fixed set of parameter values. Numerical analysis identifies five different equilibrium spatial patterns under a set of parameter values.

Next, in section 4, we calculate the total net rent which corresponds to each urban configuration. This calculation prepares us to carry out the analysis in the next section, and enables us to compare equilibrium urban configurations and optimum urban configurations.[3]

[3]Because of space limitations, in this paper we discuss only equilibrium urban configurations. However, it is shown in Ogawa–Fujita (1980) that we can formulate an optimum model of urban land use corresponding to the equilibrium model described below, and that there exists a simple relationship between the two models, which enables us to obtain optimum solutions directly from equilibrium solutions.

Finally, in section 5, we study the change in urban spatial structure within the framework of comparative static analysis. It is shown that the city must discontinuously change its spatial structure from one pattern to another pattern at critical values of the parameters. That is, catastrophic modes of structural transition are observed.

2. Formulation of the model

2.1. City

Suppose a city develops on a long narrow strip of homogeneous agricultural land of width 1 (unit distance). We assume that the width of the land is sufficiently small, and hence the city may be treated as a linear city. Each location in the city is representable by a point, x, on the line.

Economic activity in the city is assumed to be generated by two types of actors: households and business firms. The problems we consider in this paper are the interactions between activity units in the city and the resultant urban configuration of the spatial economy. That is, households supply labor to business firms, and conversely, business firms pay wages to households; such activities may be called the between-sector interactions: business firms interact each other and obtain agglomeration economies; these activities may be called the within-sector interactions. In addition, activity units in both sectors compete for land (for residential and production use); this competition involves both between-sector and within-sector interactions. These simultaneous interactions take place through labor and land markets, both of which are assumed to be perfectly competitive everywhere in the city.

2.2. Household

Suppose there are N identical households in the city. We assume they have identical preferences for land and composite commodity. The household utility function for each household is expressed by

$$U = U(S, Z), \tag{2.1}$$

where U is utility level, S is land consumed by the household, Z is composite commodity consumed by the household, and $\partial U/\partial S > 0$, $\partial U/\partial Z > 0$. Each household contains one worker supplying his (or her) labor to a business firm. The wage earned by that worker is the only income for each household. The travel of each household consists solely of the journey to work. Assuming that the composite commodity is imported from outside the city at a constant price p_z, the budget constraint of a household locating at x and working at x_w is given by

$$W(x_w) = R(x)S + p_z Z + td(x, x_w), \tag{2.2}$$

where $W(x_w)$ is wage paid by the business firms locating at x_w, $R(x)$ is land rent for a unit of land at x, $d(x, x_w) = |x - x_w|$ is distance between residence and job site, and t is commuting cost per unit of distance.

The objective of each household is to maximize its utility (2.1), subject to its budget constraint (2.2), by appropriately choosing S, Z, x and x_w. In this paper, for simplicity of analysis, we assume the lot size of each household is fixed at some positive constant size S_h. Consequently, the objective of a household is equivalent to choosing the residential location, x, and the job site, x_w, so as to maximize the consumption level of the composite commodity:

$$\max_{x, x_w} Z = \frac{1}{p_z}(W(x_w) - R(x)S_h - td(x, x_w)). \tag{2.3}$$

2.3. *Business firm*

There are M identical business firms.[4] Each business firm produces some kind of service, information or goods using land and labor as inputs, and production output is exported from the city at a constant price, p_0. We assume, again for simplicity of analysis, that the amounts of land and labor used for production by each firm are fixed at the positive constants S_b and L_b, respectively. Then, assuming that a full employment prevails in the city, the following relation must hold at the equilibrium:

$$M = N/L_b. \tag{2.4}$$

The production mode in modern cities is often characterized by the concept of agglomeration economies,[5] which is one of the main reasons for the existence and growth of cities. In this paper, agglomeration economies are considered only for the business firm sector, and are treated as follows. First, as the measure of agglomeration economies for each business firm, we introduce the *locational potential function* $F(x)$ which is defined as

$$F(x) = \int b(y)\, e^{-\alpha d(x,y)}\, dy, \tag{2.5}$$

[4]For simplicity, we assume that all business firms in the city are identical *from the viewpoint of location behavior*; but they may be different in some aspects such as, for example, contents of services or kinds of information produced by them.

[5]Broadly speaking, agglomeration economies are the potential advantages enjoyed by economic behaving units through spatial concentration of activities. For a good discussion and summary of this concept, see Kawashima (1971).

where $F(x)$ is locational potential at x, $b(y)$ is density of business firms at y, α is potential parameter ($\geqq 0$), and $d(x,y)=|x-y|$ is distance between firms locating at x and y. Next, by using this locational potential function, we assume that the behavior of each business firm can be described according to either a multiplicative production equation,

$$\max_{x} \pi = p_0 f(S_b, L_b)\beta F(x) - R(x)S_b - W(x)L_b, \tag{2.6}$$

or an additive production equation,

$$\max_{x} \pi = p_0 f(S_b, L_b) + p_p F(x) - R(x)S_b - W(x)L_b, \tag{2.7}$$

where π is profit level, β is output conversion rate of locational potential, $R(x)$ is land rent for a unit of land at x, $W(x)$ is wage for a unit of labor at x, and p_p is monetary conversion rate of locational potential.

We may interpret the multiplicative form given in (2.6) as suggesting that the effect of agglomeration economies is to raise productivity at locations of high locational potential. On the other hand, the locational advantages of side-benefits and/or cost-reductions (except land rents and wages) in production due to proximity to other business firms are capitalized in the additive form.[6] However, both representations of firm behavior are mathematically equivalent. To see this, using the assumption of constant land and labor inputs, (2.6) and (2.7) can be rewritten as follows:

$$\pi = kF(x) - R(x)S_b - W(x)L_b \quad \text{where} \quad k = p_0 f(S_b, L_b)\beta, \tag{2.8}$$

$$\pi = k' + p_p F(x) - R(x)S_b - W(x)L_b \quad \text{where} \quad k' = p_0 f(S_b, L_b). \tag{2.9}$$

If we now set $\pi - k' = \pi$ and $p_p = k$ in (2.9), then (2.9) becomes identical to (2.8). Hence, (2.6) and (2.7) are mathematically equivalent. In the following analysis, we will use the multiplicative form given in (2.8) as the description of the business firm's behavior.

[6] For example, suppose that business production requires transactions (i.e., communications or information exchange) among themselves [Capozza (1976), O'Hara (1977)]. There are two possible ways to take into account those transactions; one is to incorporate them in production function, and another is to consider them as an element of the cost or profit function. In the case of the first approach, suppose that the original production function of each business firm is given by $f_0(S_b, L_b, F(x))$, where $F(x)$ is the amount of business information available at location x. Then, if $f_0(S_b, L_b, F(x)) = f(S_b, L_b)F(x)$, we have (2.8). This is an example of *Marshallian external economies*. In the case of the second approach, suppose that the total transaction cost for a firm at location x is given by $T(x) = \int b(y)\tau(1 - e^{-\alpha|x-y|})\,dy$, where τ is a positive constant. Then, since $T(x) = \tau M - \tau F(x)$, we have $\pi = p_0 f(S_b, L_b) - T(x) - R(x)S_b - W(x)L_b = (-\tau M + p_0 f(S_b, L_b)) + \tau F(x) - R(x)S_b - W(x)L_b$, which is essentially the same with (2.9).

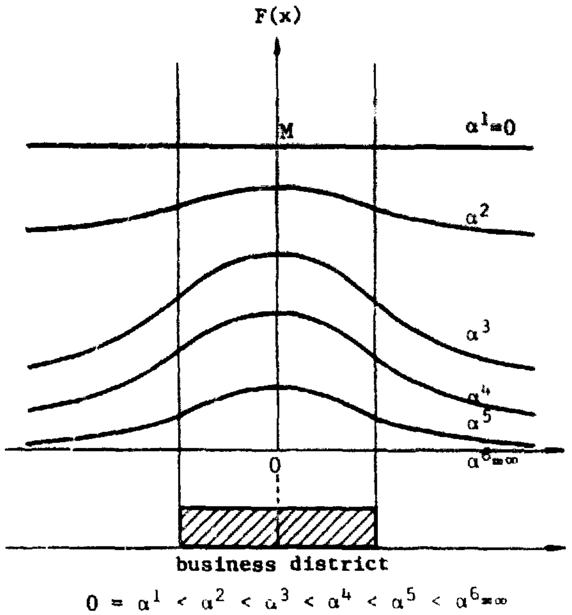

Fig. 1. Effect of agglomeration parameter α on locational potential $F(x)$: monocentric case.

Notice that in formulation (2.8) or (2.9), the distribution of all business firms in the city affects the decision of each business firm through the locational potential function $F(x)$. For this reason, it is important to understand the functional characteristics of $F(x)$, in particular, the effects of the potential parameter α. From the definition, it is obvious that $F(x)=M$ (the total number of business firms) when $\alpha=0$, and $F(x)=0$ when $\alpha=\infty$. In other words, the locational potential of each location is a maximum at $\alpha=0$ and is monotonically decreasing with α; this is true for any distributional pattern of business firms. Fig. 1 shows the change of locational potential at various values of α in the case of monocentric uniform distribution of business firms. Note that locational potential differentials, the difference of potential levels among locations, are greatest when α takes a medium value. And, the locational decision of each business firm is affected by the magnitude of locational potential *differentials*. It must also be noted that locational potential differentials are dependent not only on the value of potential parameter α, but also on the distribution of business firms over the space.

2.4. Equilibrium conditions

In the context described in the previous sections, our task is to analyze the equilibrium spatial structure of the city. As noted before we assume that the population of the city is fixed at N; however, firms are free to enter or leave the city. Then, at the equilibrium, competition drives the profit level of each firm to zero, and the total number M of firms in the city is given by (2.4). Hence, each equilibrium spatial structure of the city is described by a system, $\{h(x), b(x), R(x), W(x), P(x, x_w), U\}$ where $h(x)$ is household density function, $b(x)$ is business firm density function, $R(x)$ is land rent profile, $W(x)$ is wage profile, $P(x, x_w)$ is number of households locating at x and commuting to x_w divided by the total number of households locating at x: commuting pattern, and U is utility level.

To state the equilibrium conditions for the problem, we define the following functions:

$$\Psi(x) \equiv \Psi(x \mid \boldsymbol{W}(x_w), U) = \max_{x_w} \left\{ \frac{1}{S_h} (W(x_w) - p_z Z - td(x, x_w)) \mid U(S_h, Z) = U \right\}, \tag{2.10}$$

$$\Phi(x) \equiv \Phi(x \mid \boldsymbol{W}(x), \boldsymbol{b}(x), \pi) = \frac{1}{S_b} (kF(x) - \pi - W(x) L_b), \tag{2.11}$$

where $\boldsymbol{W}(x)$ is the wage profile over all x, $W(x)$ is the specific value of $\boldsymbol{W}(x)$ at x, $\boldsymbol{b}(x)$ is the distribution of business firms over all x. $\Psi(x)$ gives the maximum land rent which could be paid by a household at location x while deriving the utility level U, given the wage profile $\boldsymbol{W}(x_w)$. Similarly, $\Phi(x)$ is the maximum land rent which a business form could pay at location x while deriving the profit level π, given the distribution of business firms $\boldsymbol{b}(x)$. We call $\Psi(x)$ and $\Phi(x)$, the *bid rent function of household* and *bid rent function of business firm*, respectively. These functions are generalized forms of the bid rent function originally defined by Alonso (1964) in the context of monocentric urban land use.

Then, the necessary and sufficient conditions for a system $\{h(x), b(x), R(x), W(x), P(x, x_w), U^*\}$ to be an equilibrium configuration are summarized as follows:

(i) land market equilibrium condition: at each x,

(i.1) $R(x) = \max\{\Psi^*(x), \Phi^*(x), R_A\}$,

(i.2) $R(x) = \Psi^*(x)$ if $h(x) > 0$,

(i.3) $R(x) = \Phi^*(x)$ if $b(x) > 0$,

(i.4) $R(x) = R_A$ on the urban fringe,

(i.5) $S_h h(x) + S_b b(x) \leq 1$,

(i.6) $S_h h(x) + S_b b(x) = 1$ if $R(x) > R_A$,

where $\Psi^*(x) = \Psi(x \mid W(x_w), U^*)$,

$\Phi^*(x) = \Phi(x \mid W(x), h(x), \pi = 0)$,

R_A = agricultural land rent (exogenously given),

(ii) labor market equilibrium condition: at each x

$$b(x) L_b = \int h(y) P(y, x)\,\mathrm{d}y,$$

(iii) total unit number constraints

$$\int h(x)\,\mathrm{d}x = N, \qquad \int b(x)\,\mathrm{d}x = N/L_b,$$

(iv) non-negativity constraints

$$h(x) \geqq 0, \quad b(x) \geqq 0, \quad R(x) \geqq 0,$$

$$W(x) \geqq 0, \quad 1 \geqq P(x, x_w) \geqq 0,$$

$$\int P(x, x_w)\,\mathrm{d}x_w = 1.$$

Conditions (i.1) to (i.3) simply mean that, at each location, land is occupied by the maximum bidder (under equilibrium U^* or $\pi = 0$) for the land.

3. Equilibrium urban configurations

3.1. Numerical approach

In this section, we explore the spatial structures of solutions for the equilibrium model. Because of the complicated functional form of the locational potential function $F(x)$ (although it seems simple at first glance), it is quite difficult to manipulate our model analytically. Hence, after analyzing the conditions of solutions in terms of parameters, we carry out numerical explorations to investigate the specific properties of solutions.

To do so, a specification of values of parameters is required. The parameters governing the character of solutions are: production parameter k, commuting rate t, potential parameter α, total number of households N, lot size of each household S_h, land input of production S_b, and labor input of production L_b. Among these parameters, k, t and α are especially important. Therefore, in the following we arbitrarily fix the values of N, S_h, S_b and L_b as follows:

$$\{N, S_h, S_b, L_b\} = \{1000, 0.1, 1, 10\},$$

and the results of analysis are shown on $\{k, t, \alpha\}$-space. However, as we will see later, $\{k, t, \alpha\}$-space can be collapsed into $\{t/k, \alpha\}$-space. This implies that the properties of solutions are essentially governed by two main factors: the ratio of the commuting rate to the production parameter and the potential parameter. The effect of changes in values of other parameters are briefly discussed in section 3.4. We introduce the following terminologies and notations for convenience in the subsequent argument:

(i) (exclusive) residential area: $RA = \{x \mid h(x) > 0, b(x) = 0\}$,

(ii) (exclusive) business area: $BD = \{x \mid h(x) = 0, b(x) > 0\}$,

(iii) integrated district: $ID = \{x \mid h(x) > 0, b(x) > 0\}$.

3.2. *Equilibrium urban configuration*

In this section, we first examine the possibility of those types of non-monocentric urban configurations which have been observed in Ogawa and Fujita (1978). Next, multicentric urban configurations are analyzed; in particular, we investigate closely two kinds of multicentric urban configurations: duocentric and tricentric. All other urban configurations with more than three centers are left for future investigation. We also limit our analysis to the case of symmetric urban configurations.

3.2.1. *Monocentric urban configuration*

Take the origin to be the center of the city [refer to fig. 2(a)]. The density functions of households and business firms are given, respectively, by

$$h(x) = \frac{1}{S_h}, \qquad b(x) = 0 \quad \text{for } x \in RA,$$

$$h(x) = 0, \quad b(x) = \frac{1}{S_b} \qquad \text{for } x \in BD.$$

From these density functions and from the total unit number constraints, the boundary, f_1, between BD and RA and the urban fringe, f_2, are obtained as follows:

$$f_1 = \frac{S_b N}{2L_h}, \qquad f_2 = \frac{S_b + S_h L_b}{2L_b} N.$$

By the assumption of symmetry in urban configuration, it suffices to examine the equilibrium conditions on the right half of the city where $x \geq 0$. No cross-commuting should exist at the equilibrium; and hence, the

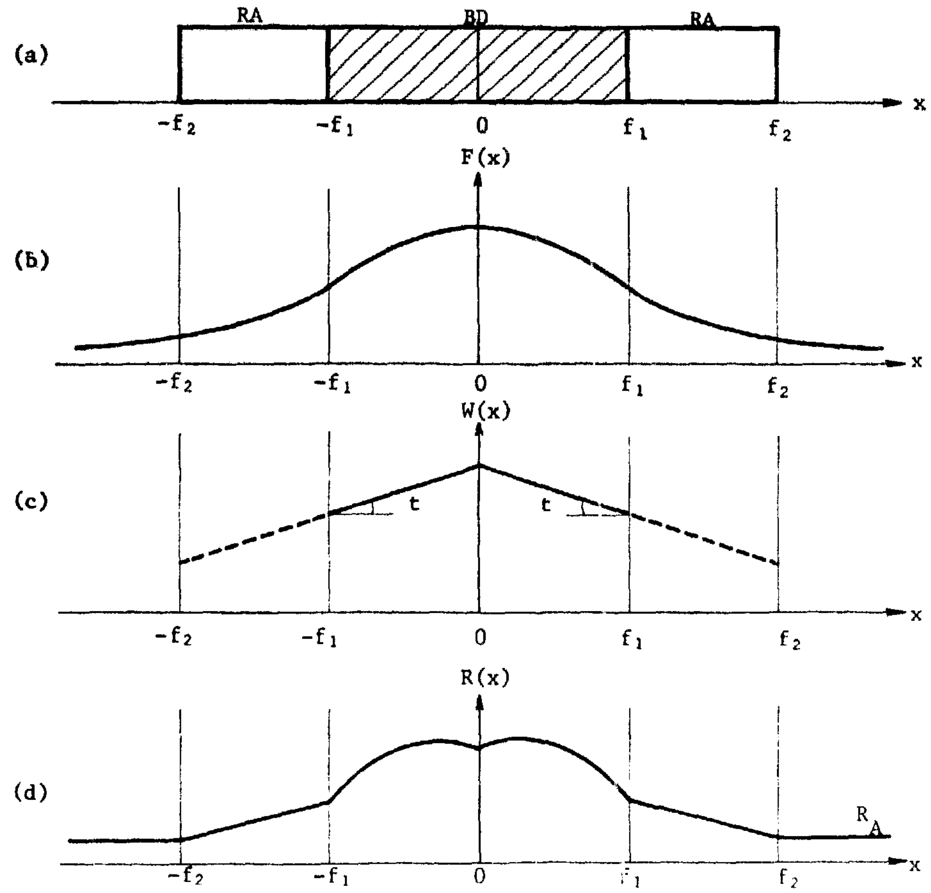

Fig. 2. Monocentric urban configuration.

equilibrium wage profile is given by[7]

$$W(x)=W(0)-tx, \tag{3.1}$$

where $W(0)$ is the wage paid by business firms at the origin, and $W(x)$ is the wage paid at location x if $x\in BD$ and the disposable income for households at x if $x\in RA$, since $W(x_w)-td(x,x_w)=W(0)-tx_w-td(x,x_w)=W(0)-tx=W(x)$. This wage profile is depicted in fig. 2(c).

Next, the locational potential function is written as follows:

$$F(x)=\int_{-f_1}^{f_1} b(y)e^{-\alpha|x-y|}\mathrm{d}y$$

$$=\frac{1}{\alpha S_b}(2-e^{-\alpha(f_1+x)}-e^{-\alpha(f_1-x)}), \quad \text{for} \quad x\in[0,f_1], \tag{3.2a}$$

[7]The validity of (3.1) and no cross-commuting at the equilibrium is intuitively clear. For rigorous proof of this point, see properties 1 and 2 of Ogawa–Fujita (1978).

$$F(x)=\frac{1}{S_b}\int_{-f_1}^{f_1} e^{-\alpha(x-y)}\,dy$$

$$=\frac{1}{\alpha S_b}(e^{-\alpha(x-f_1)}-e^{-\alpha(x+f_1)}).$$

$$\text{for} \quad x\in[f_1,\infty], \tag{3.2b}$$

From (3.2), when $0<\alpha<\infty$, function $F(x)$ is monotonically decreasing, concave on BD and convex on RA, as depicted in fig. 2(b).

Given $W(x)$ and $F(x)$ from (3.1) and (3.2), the equilibrium conditions in the land market are rewritten as follows:

$$R(x)=\Phi^*(x)\geqq\Psi^*(x) \qquad \text{for } x\in[0,f_1], \tag{3.3}$$

$$R(x)=\Phi^*(x)=\Psi^*(x) \qquad \text{at } x=f_1, \tag{3.4}$$

$$R(x)=\Psi^*(x)\geqq\Phi^*(x) \qquad \text{for } x\in[f_1,f_2], \tag{3.5}$$

$$R(x)=\Psi^*(x)=R_A \qquad \text{at } x=f_2, \tag{3.6}$$

$$\Psi^*(x)=\frac{1}{S_h}(W(x)-p_zZ^*), \tag{3.7}$$

$$\Phi^*(x)=\frac{1}{S_b}(kF(x)-W(x)L_b), \tag{3.8}$$

where Z^* denotes the amount of the composite good consumed at the equilibrium, i.e.,

$$U^*=U(S_h,Z^*). \tag{3.9}$$

Since $\Psi(x)$ is linear and $\Phi(x)$ is concave on BD and convex on RA, the above conditions are equivalent to the following conditions:

$$R(0)=\Phi^*(0)\geqq\Psi^*(0),$$

$$R(f_1)=\Phi^*(f_1)=\Psi^*(f_1),$$

$$R(f_2)=R_A=\Psi^*(f_2)\geqq\Phi^*(f_2),$$

which imply

$$\Phi^*(0)-\Phi^*(f_1)\geqq\Psi^*(0)-\Psi^*(f_1),\quad \text{i.e.,}$$

$$[k(F(0)-F(f_1))-f_1tL_b]/S_b\geqq f_1t/S_h, \tag{3.10}$$

$$\Phi^*(f_1)-\Phi^*(f_2)\geqq\Psi^*(f_1)-\Psi^*(f_2),\quad \text{i.e.,}$$

$$[k(F(f_1)-F(f_2))-(f_2-f_1)tL_b]/S_b\geqq(f_2-f_1)t/S_h. \tag{3.11}$$

Hence, we finally obtain

$$t\leqq kK\frac{F(0)-F(f_1)}{f_1}, \tag{3.12}$$

$$t\leqq kK\frac{F(f_1)-F(f_2)}{f_2-f_1}, \tag{3.13}$$

or equivalently

$$\frac{t}{k}\leqq\min\left\{\frac{F(0)-F(f_1)}{f_1},\ K\frac{F(f_1)-F(f_2)}{f_2-f_1}\right\}, \tag{3.14}$$

where $K=S_h/(S_b+S_hL_b)$. Under the wage profile given by (3.1), any commuting pattern satisfying condition (ii) and the property of no cross-commuting is consistent with equilibrium. The corresponding equilibrium land rent profile $R(x)$ is uniquely determined from (3.6), (3.7) and (3.8) as follows:

$$\begin{aligned} R(x)&=\frac{k}{S_b}(F(x)-F(f_1))-\frac{L_b}{S_b}(f_1-x)t+\frac{1}{S_h}(f_2-f_1)t\\ &\quad+R_A \quad \text{for} \quad x\in[0,f_1],\\ &=\frac{1}{S_h}(f_2-x)t+R_A \quad \text{for} \quad x\in[f_1,f_2],\\ &=R_A \quad \text{for} \quad x\in[f_2,\infty]. \end{aligned} \tag{3.15}$$

Therefore, we conclude that the monocentric urban configuration is an equilibrium if and only if conditions (3.6) and (3.14) are satisfied. That is, as we can see from (3.10) and (3.11), the monocentric urban configuration can be an equilibrium only when locational potential differentials are sufficiently

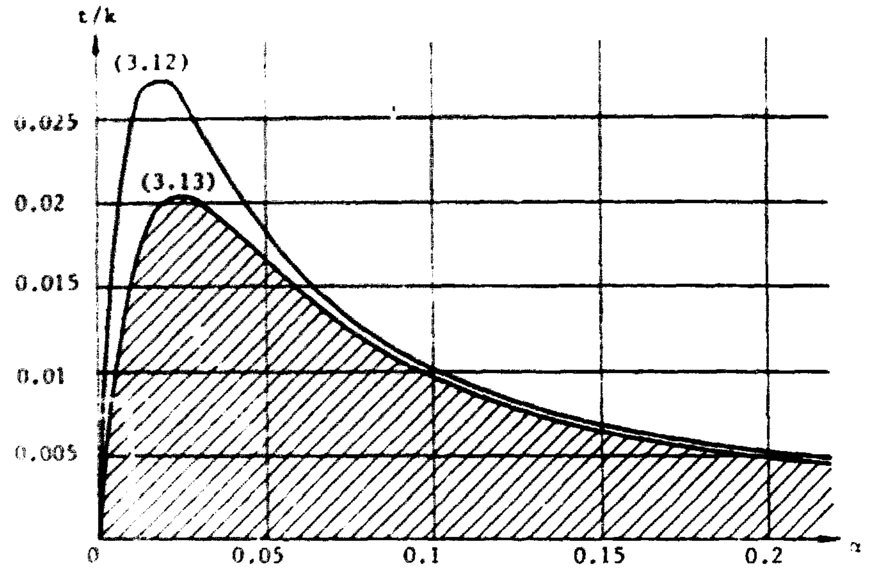

Fig. 3. Equilibrium condition on $\{t/k, \alpha\}$ for monocentric urban configuration.

large compared with commuting cost rate t, i.e., only when t is sufficiently small compared with locational potential differentials.

The result of numerical analysis on condition (3.14) is summarized in fig. 3. As seen from the figure, condition (3.12) is always satisfied when (3.13) is satisfied, and hence (3.14) can be reduced to (3.13). If a combination of parameters t/k and α lies in the shaded area of $\{t/k, \alpha\}$-space, then the monocentric urban configuration is a solution under that combination, $(t/k, \alpha)$.[8] Observe from fig. 3 that when $\alpha \doteqdot 0.025$, the monocentric urban configuration can be an equilibrium under the greatest range of t/k values. This occurs as locational potential differentials are maximum when $\alpha \doteqdot 0.025$.

3.2.2. Completely mixed urban configuration

Next, suppose households and business firms coexist at every location in the city [see fig. 4(a)], and that commuting does not exist in the city. Then, the density functions of households and business firms are

$$h(x) = \frac{L_b}{S_b + S_h L_b}, \quad b(x) = \frac{1}{S_b + S_h L_b} \quad \text{for} \quad x \in [-f_1, f_1],$$

where f_1 and $-f_1$ denote the urban fringes. From the total unit number conditions, we get $f_1 = ((S_b + S_h L_b)/2L_b)N$.

[8]Whenever the value of production parameter k is sufficiently large compared with the value of R_A condition (3.6) can be always satisfied when (3.14) is satisfied. Therefore, in the following discussion we implicitly assume that condition (3.6) (or the equivalent one) is always satisfied and we do not examine it.

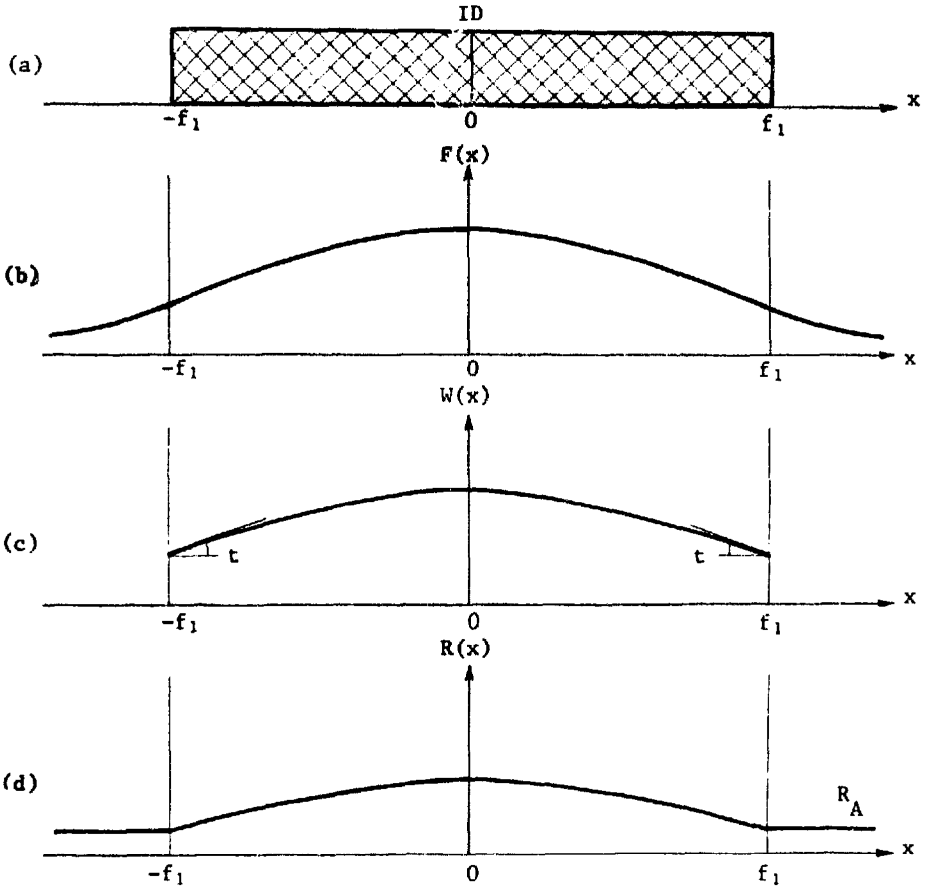

Fig. 4. Completely mixed urban configuration.

Since the equilibrium condition in the labor market is satisfied by the assumption of no commuting pattern, i.e., $P(x, x_w) = 1$ for $x \in [-f_1, f_1]$, $x_w \in [-f_1, f_1]$, we can move on to an examination of the equilibrium conditions in the land market. They are

$$R(x) = \Psi^*(x) = \Phi^*(x) \quad \text{for } x \in [-f_1, f_1], \tag{3.16}$$

$$R(x) = R_A \quad \text{at } x = -f_1, f_1, \tag{3.17}$$

where $\Phi^*(x)$ is given by (3.8), and $\Psi^*(x)$ must be given by (3.7) since the assumption of no commuting implies $x_w = x$. From (3.7), (3.8) and (3.16), we obtain

$$W(x) = \frac{1}{S_b + S_h L_b} \{S_h k F(x) + S_b p_z Z^*\}, \tag{3.18}$$

$$R(x)=\frac{1}{S_b+S_hL_b}\{kF(x)-L_bp_zZ^*\}$$
$$=\frac{k}{S_b+S_hL_b}(F(x)-F(f_1))+R_A. \tag{3.19}$$

The locational potential function for the completely mixed urban configuration is given by

$$F(x)=\frac{1}{\alpha(S_b+S_hL_b)}(2-e^{-\alpha(f_1+x)}-e^{-\alpha(f_1-x)}) \quad \text{for} \quad x\in[-f_1,f_1],$$
$$=\frac{1}{\alpha(S_b+S_hL_b)}(e^{-\alpha(x-f_1)}-e^{-\alpha(x+f_1)})$$
$$\text{for} \quad x\in[f_1,\infty],[-\infty,-f_1]. \tag{3.20}$$

Since $F'(x)\leqq 0$ (with equality holding at $x=0$) and $F''(x)<0$ for $x\in[-f_1,f_1]$, $F(x)$ is concave on ID, as shown in fig. 4(b). Consequently, the equilibrium wage profile $W(x)$ and the equilibrium land rent profile $R(x)$ are also concave functions on ID [see fig. 4(c) and 4(d)].

Finally, no commuting implies that $|W'(x)|\leqq t$ for all $x\in[-f_1,f_1]$, which is equivalent to $W'(f_1)\geqq -t$ because of the strict concavity of $W(x)$ and the symmetry of the urban configuration. From this condition, we get

$$\frac{t}{k}\geqq\frac{S_h}{(S_b+S_hL_b)^2}(1-e^{-2\alpha f_1}). \tag{3.21}$$

Accordingly, the completely mixed urban configuration is an equilibrium solution if and only if (3.17) and (3.21) are satisfied, that is commuting cost rate t is sufficiently large compared with the locational potential differentials given by (3.20). Fig. 5 illustrates condition (3.21) under the values of the parameters given in section 3.1.

3.2.3. Incompletely mixed urban configuration

An incompletely mixed land use pattern is characterized by the following density functions and boundaries; for $x\geqq 0$

$$h(x)=\frac{L_b}{S_b+S_hL_b}, \qquad b(x)=\frac{1}{S_b+S_hL_b} \quad \text{for } x\in[0,f_1],$$
$$h(x)=0, \qquad b(x)=1/S_b \qquad \text{for } x\in[f_1,f_2],$$

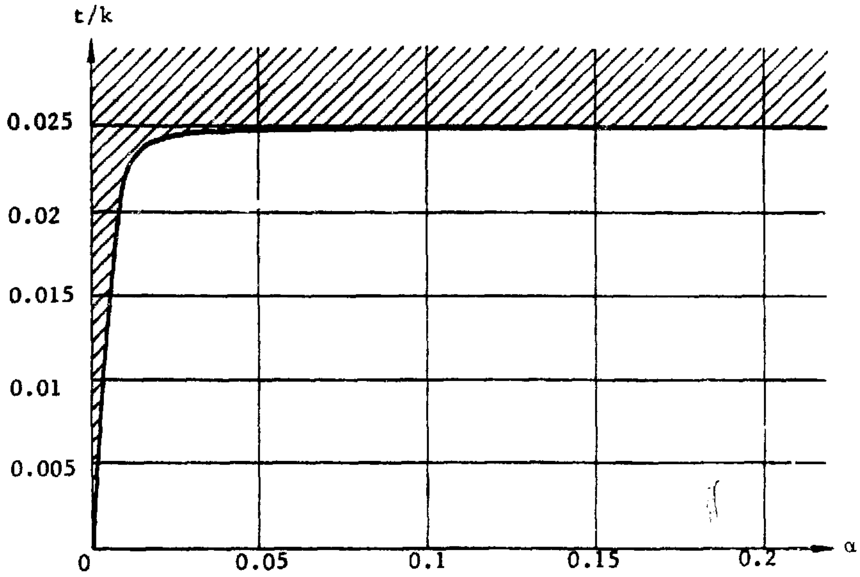

Fig. 5. Equilibrium condition on $\{t/k, \alpha\}$ for completely mixed urban configuration.

$$h(x)=1/S_h, \qquad b(x)=0 \qquad \text{for } x\in[f_2,f_3], \quad \text{and}$$

$$f_1\in\left(0, \frac{S_b+S_hL_b}{2L_b}N_h\right), \quad f_2=\frac{S_hL_b}{S_b+S_hL_b}f_1+\frac{S_bN}{2L_b}, \quad f_3=\frac{S_b+S_hL_b}{2L_b}N.$$

Fig. 6 shows one example of an incompletely mixed pattern. Obviously, this pattern approaches the monocentric pattern as f_1 approaches 0, and approaches the completely mixed pattern as f_1 approaches $((S_b+S_hL_b)/2L_b)N$.

The equilibrium conditions in the land market for the incompletely mixed urban configuration are summarized, for $x\geqq 0$, as follows:

$$R(x)=\Psi^*(x)=\Phi^*(x) \qquad \text{for } x\in[0,f_1], \tag{3.22}$$

$$R(x)=\Phi^*(x)\geqq\Psi^*(x) \qquad \text{for } x\in[f_1,f_2], \tag{3.23}$$

$$R(x)=\Psi^*(x)=\Phi^*(x) \qquad \text{at } x=f_2, \tag{3.24}$$

$$R(x)=\Psi^*(x)\geqq\Phi^*(x) \qquad \text{for } x\in[f_2,f_3], \tag{3.25}$$

$$R(x)=\Psi^*(x)=R_A \qquad \text{at } x=f_3. \tag{3.26}$$

where $\Psi^*(x)$ and $\Phi^*(x)$ are given by (3.7) and (3.8), respectively. As in the case of a completely mixed urban configuration, from (3.7), (3.8) and (3.22), we obtain (3.18) and (3.19) for $x\in ID$. Thus, equilibrium profiles $W(x)$ and $R(x)$ depend on the locational potential function $F(x)$. Next, since we can

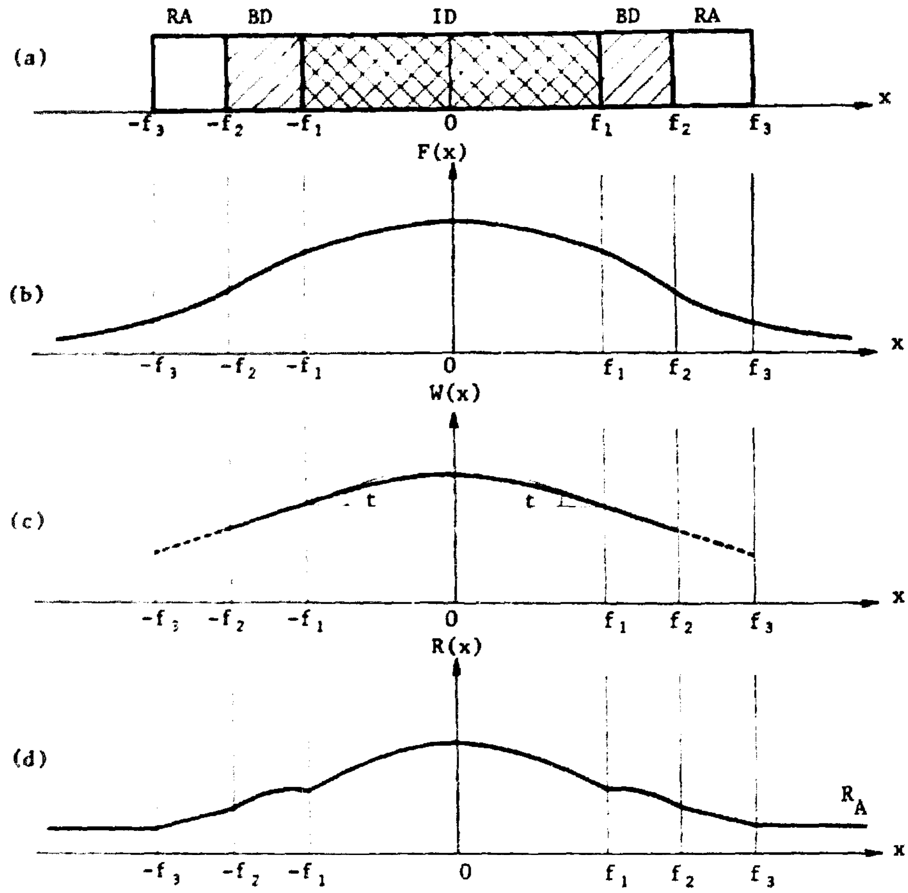

Fig. 6. Incompletely mixed urban configuration.

show that $F(x)$ is strictly concave on BD and strictly convex on RA, the rest of the conditions are equivalent to

$$R(x)=\Phi^*(x)=\Psi^*(x) \qquad \text{at } x=f_1, f_2,$$

$$R(x)=R_A=\Psi^*(x)\geqq\Phi^*(x) \quad \text{at } x=f_3.$$

From these conditions, we derive

$$\frac{t}{k}=K\frac{F(f_1)-F(f_2)}{f_2-f_1}, \tag{3.27}$$

$$\frac{t}{k}\leqq K\frac{F(f_1)-F(f_3)}{f_3-f_1}, \tag{3.28}$$

where $K=S_h/(S_h+S_b L_b)$. Finally, no commuting in ID implies that $|W'(x)|\leqq t$

for $x \in [0, f_1]$, which is equivalent to the following condition:

$$t/k \geqq K|F'(f_1)|. \tag{3.29}$$

Accordingly, it is concluded that the incompletely mixed urban configuration is an equilibrium solution if and only if conditions (3.26), (3.27), (3.28) and (3.29) are satisfied. The associated equilibrium land rent curve $R(x)$ can be depicted as in fig. 6(d). There is no commuting in *ID*, and households in *RA* commute to firms in *BD*.

In fig. 7 the value of t/k which sustains the equilibrium incompletely mixed urban configuration is depicted. The dotted lines show the equilibrium condition on t/k for specific values of f_1. As seen in the figure, the value of t/k for any given f_1 increases rapidly at first, reaches its maximum and then falls gradually as α increases. Note that there is an upper limit on α for any given f_1. For example, $\alpha \doteqdot 0.143$ when $f_1 = 80$. Moreover, it must be noted that there can exist two equilibrium solutions in the cross-hatched region in fig. 7. For example, select a point A; $(t/k, \alpha) = (0.016, 0.05)$ in that region. When two parameters are given exogenously at A the city can take two different incompletely mixed urban configurations having different sizes of integrated district $(= 2f_1)$ of either 120 or 2.

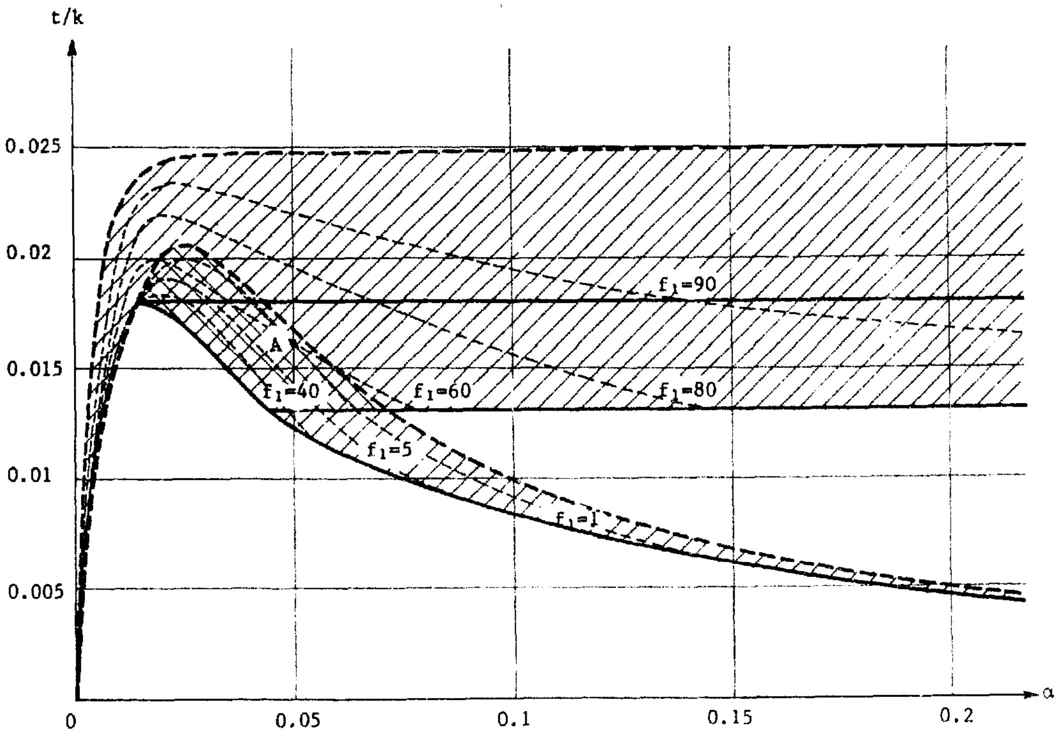

Fig. 7. Equilibrium condition on t/k for incompletely mixed urban configuration.

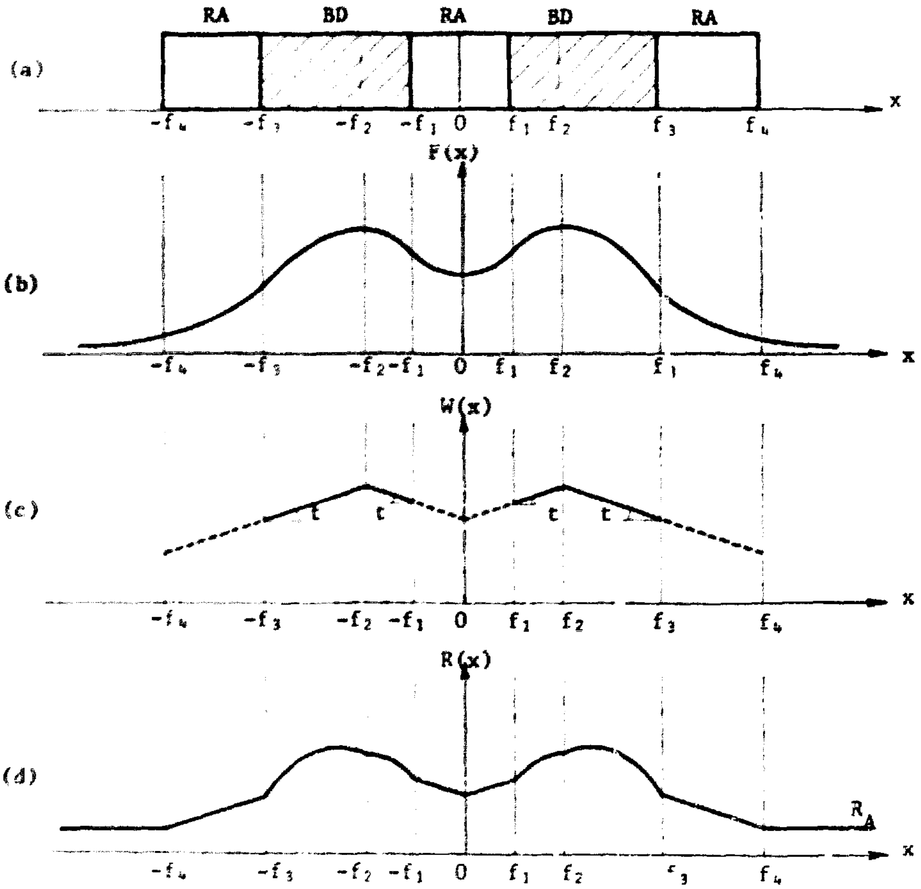

Fig. 8. Duocentric urban configuration

3.2.4. Duocentric urban configuration

We next consider the duocentric urban configuration in which business firms concentrate and form employment centers at two distinct areas. An example of a duocentric urban configuration is illustrated in fig. 8(a). From the property of no cross-commuting at the equilibrium, households between $x=0$ and f_1 commute to firms between f_1 and f_2, and households between f_3 and f_4 commute to firms between f_2 and f_3, where f_1, f_3 are boundaries between RA and BD, f_4 is urban fringe, and f_2 is the location at which business firms are divided according to rightward and leftward commuting. Then, from the property of the equilibrium wage profile, the wage is at maximum at f_2 and decreases at a constant rate t, as shown in fig. 8(c).

The density functions and boundaries, for $x \geqq 0$, are given by

$$h(x)=\frac{1}{S_h}, \quad b(x)=0 \quad \text{for} \quad x\in[0,f_1], \quad x\in[f_3,f_4],$$

$$h(x)=0, \quad b(x)=\frac{1}{S_b} \quad \text{for} \quad x\in[f_1,f_3]. \tag{3.30}$$

where

$$f_1 \in \left(0, \frac{S_h N}{4}\right), \qquad f_2 = \frac{S_b + S_h L_b}{S_h L_b} f_1,$$

$$f_3 = f_1 + \frac{S_b N}{2L_b}, \qquad f_4 = \frac{S_b + S_h L_b}{2L_b} N.$$

Thus, the duocentric land use pattern can be uniquely specified by f_1 alone since f_2 and f_3 are functions of f_1. When f_1 approaches 0, the duocentric urban configuration approaches the monocentric urban configuration. It is not difficult to show that, at the equilibrium, the duocentric pattern cannot contain agricultural land inside the city.

Once the value of f_1 is specified, the locational potential function $F(x)$ is obtainable from the definition given by (2.5), and the following conditions in land market must be satisfied:

$$R(x) = \Psi^*(x) \geqq \Phi^*(x) \quad \text{for } x \in [0, f_1], \quad x \in [f_3, f_4], \tag{3.31}$$

$$R(x) = \Psi^*(x) = \Phi^*(x) \quad \text{at } x = f_1, f_3, \tag{3.32}$$

$$R(x) = \Phi^*(x) \geqq \Psi^*(x) \quad \text{for } x \in [f_1, f_3], \tag{3.33}$$

$$R(x) = \Psi^*(x) = R_A \qquad \text{at } x = f_4, \tag{3.34}$$

where $\Psi^*(x)$ and $\Phi^*(x)$ are given by (3.7) and (3.8), respectively. From these conditions, we obtain the following set of conditions on t/k:

$$\frac{t}{k} = K \frac{F(f_1) - F(f_3)}{f_1 + f_3 - 2f_2},$$

$$\frac{t}{k} \leqq \min\left\{K \frac{F(f_2) - F(f_1)}{f_2 - f_1}, K \frac{F(f_1) - F(0)}{f_1}, K \frac{F(f_1) - F(f_4)}{f_1 + f_4 - 2f_2}\right\}, \tag{3.35}$$

where $K = S_h/(S_b + S_h L_b)$.

Accordingly, the duocentric urban configuration is an equilibrium solution if and only if conditions (3.34) and (3.35) are satisfied for some $f_1 \in (0, S_h N/4)$. The equilibrium land rent profile, $R(x)$, for this pattern can be depicted as in fig. 8(d).

The duocentric pattern can be interpreted either as one city with two business districts or as two adjoining cities creating external economies for each other and enjoying agglomeration economies within a system of cities. The existence of such a duocentric urban configuration as an equilibrium

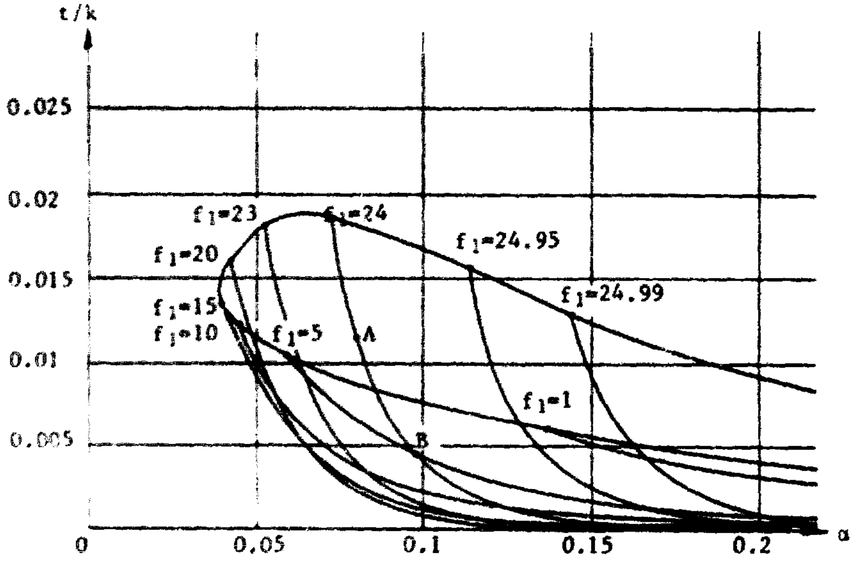

Fig. 9. Equilibrium condition on t/k for duocentric urban configuration.

solution, or equivalently, the existence of parameters which satisfy the equilibrium conditions for a duocentric pattern, is verifiable by numerical analysis. By specifying the value of f_1, condition (3.35) can be plotted in $\{t/k, \alpha\}$-space. Fig. 9 illustrates the conditions of t/k and α simultaneously for some representative values of f_1. For example, consider two points, A and B, in fig. 9. If point A represents the actual values of parameters, then the duocentric pattern can be an equilibrium pattern only when $f_1 = 24$. But, if B represents the actual values, then the duocentric pattern can be an equilibrium pattern when either $f_1 = 24$ or $f_1 = 5$. This means that the equilibrium problem has multiple solutions under a specific parameter set, as in the case of the incompletely mixed urban configuration.

3.2.5. *Tricentric urban configuration*

Finally, let us consider the urban configuration associated with three centers of business districts, which may be called the tricentric urban configuration. Two cases are possible: (i) type A where all workers commute inwardly, and (ii) type B where a city is divided into three subcities with respect to the supply and demand of labor. Figs. 10(a) and 11(a) illustrate these two types of tricentric patterns. In type A, all workers living between f_1 and f_2 commute to business firms located between 0 and f_1; a portion of workers living between f_3 and f_4 commutes to business firms located between f_2 and f_3 and the rest of workers commute the much longer distance to the BD at the center. On the other hand, in type B, the city is divided by boundary points f_2 and $-f_2$ into three parts and each part has its own BD

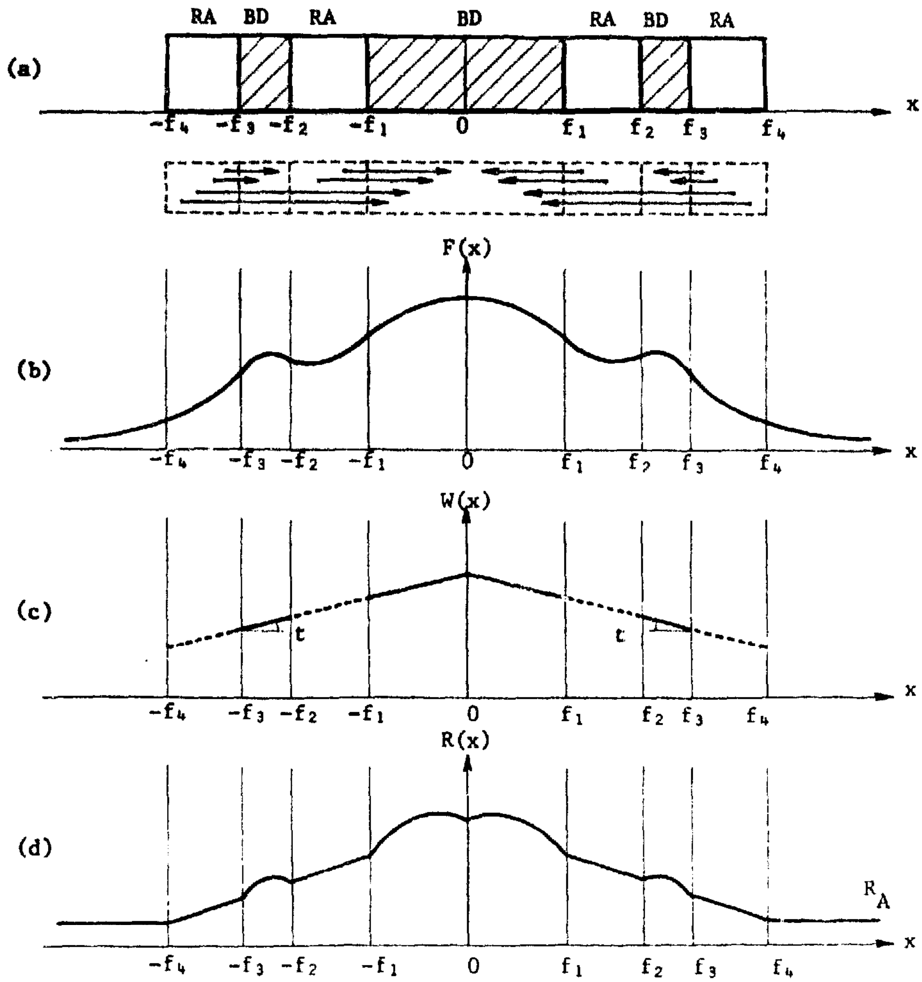

Fig. 10. Tricentric urban configuration: Type A.

whose labor for production is supplied only by households located within its domain.

The density functions of household and business firm, for $x \geqq 0$, are given by

type A

$$h(x)=0, \quad b(x)=\frac{1}{S_b} \quad \text{for } x\in[0,f_1], \quad x\in[f_2,f_3],$$

$$h(x)=\frac{1}{S_h}, \quad b(x)=0 \quad \text{for } x\in[f_1,f_2], \quad x\in[f_3,f_4],$$

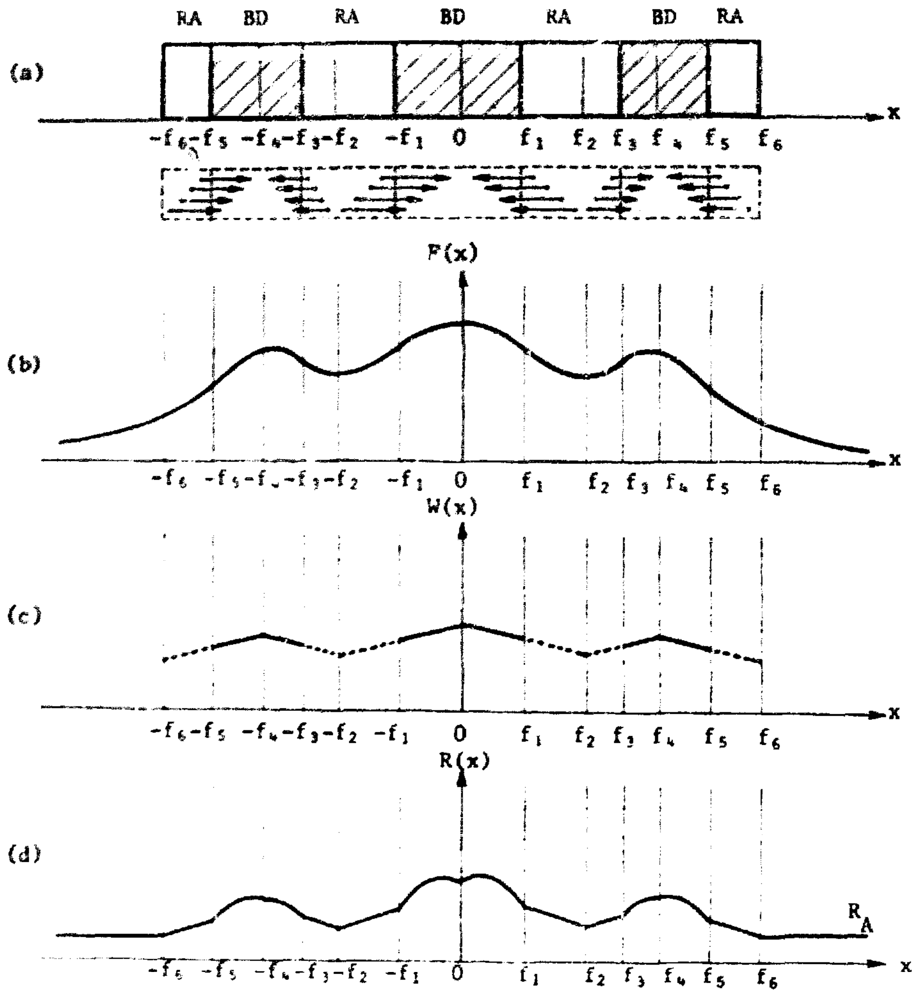

Fig. 11. Tricentric urban configuration: Type B.

where

$$f_1 \in \left(0, \frac{S_b N}{2L_b}\right), \qquad f_2 \in \left(f_1, \frac{S_b + S_h L_b}{S_b} f_1\right],$$

$$f_3 = f_2 - f_1 + \frac{S_b N}{2L_b}, \qquad f_4 = \frac{S_b + S_h L_b}{2L_b} N.$$

Type B

$$h(x) = 0, \quad b(x) = \frac{1}{S_b} \quad \text{for} \quad x \in [0, f_1], \quad x \in [f_3, f_5],$$

$$h(x) = \frac{1}{S_h}, \quad b(x) = 0 \quad \text{for} \quad x \in [f_1, f_3], \quad x \in [f_5, f_6].$$

where

$$f_1 \in \left(0, \frac{S_b N}{2L_b}\right), \qquad f_2 = \frac{S_b + S_h L_b}{S_b} f_1,$$

$$f_3 \in \left(\frac{S_b + S_h L_b}{S_b} f_1, \frac{2S_b + S_h L_b}{2S_b} f_1 + \frac{S_h N}{4}\right], \qquad f_4 = \frac{S_b + S_h L_b}{S_h L_b}(f_3 - f_1),$$

$$f_5 = f_3 - f_1 + \frac{S_b N}{2L_b}, \qquad f_6 = \frac{S_b + S_h L_b}{2L_b} N.$$

The locational potential profile, $F(x)$, and wage profile, $W(x)$, for each type, are depicted in figs. 10(b) and 10(c), 11(b) and 11(c), respectively. Hence, the following conditions must be satisfied in the land market:

$$R(x) = \Phi^*(x) \geqq \Psi^*(x) \quad \text{for } x \in BD, \tag{3.36}$$

$$R(x) = \Psi^*(x) \geqq \Phi^*(x) \quad \text{for } x \in RA, \tag{3.37}$$

$$R(x) = \Psi^*(x) = \Phi^*(x) \quad \text{at boundaries between } BD \text{ and } RA, \tag{3.38}$$

$$R(x) = \Psi^*(x) = R_A \quad \text{at urban fringe.} \tag{3.39}$$

From these conditions, the equilibrium conditions on t/k are summarized as follows:

type A

$$\frac{t}{k} = K\frac{F(f_1) - F(f_2)}{f_2 - f_1} = K\frac{F(f_1) - F(f_3)}{f_3 - f_1},$$

$$\frac{t}{k} \leqq \min\left\{K\frac{F(0) - F(f_1)}{f_1}, \frac{F(f_1) - F(f_4)}{f_4 - f_1}\right\}, \tag{3.40}$$

type B

$$\frac{t}{k} = K\frac{F(f_1) - F(f_3)}{2f_2 - f_1 - f_3} = K\frac{F(f_1) - F(f_4)}{f_5 + 2f_2 - f_1 - 2f_4},$$

$$\frac{t}{k} \leqq \min\left\{K\frac{F(0) - F(f_1)}{f_1}, K\frac{F(f_1) - F(f_2)}{f_2 - f_1},\right.$$

$$\left. K\frac{F(f_1) - F(f_4)}{2f_2 - f_1 - f_4}, K\frac{F(f_1) - F(f_6)}{2f_2 + f_6 - f_1 - 2f_4}\right\}. \tag{3.41}$$

where $K = S_h/(S_b + S_h L_b)$.

Consequently, the tricentric urban configuration is an equilibrium solution if and only if conditions (3.39) and (3.40) are satisfied for type A and conditions (3.39) and (3.41) are satisfied for type B. The equilibrium land rent profile, $R(x)$, can be depicted as in figs. 10(d) and 11(d), respectively. We can consider the tricentric pattern of type A as the spatial system of one city with a central business district and two subcenters. On the other hand, the tricentric pattern of type B may be regarded either as a system of cities, in which each city has its own CBD, or, as one city with three subcenters.

The main variables which specify the land use pattern are f_1 and f_2 for type A, f_1 and f_3 for type B. As explained above, the feasible domains of f_2 and f_3 are determined by the choice of f_1. Accordingly, in the following numerical analysis, we first fix the value of f_1, second choose the values of f_2 and f_3 in the feasible domains, and finally check whether the equilibrium conditions on t/k, given by (3.40) and (3.41), are satisfied under various values of α. Figs. 12 and 13 summarize the results.[9] The main differences between

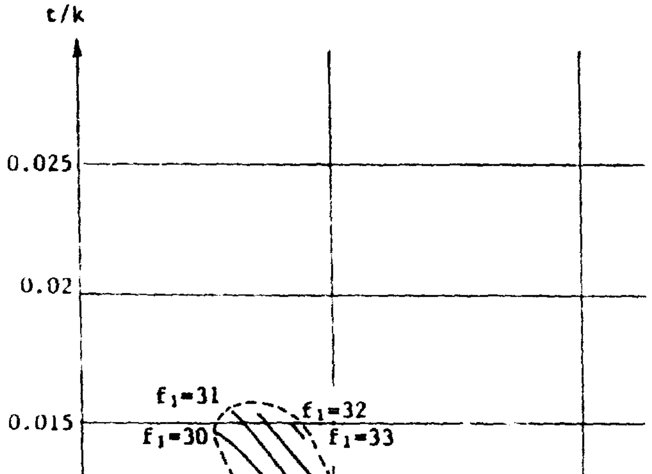

Fig. 12. Equilibrium condition on t/k for tricentric urban configuration of type A.

[9] For the details of the results on numerical calculation for tricentric urban configurations, see Ogawa and Fujita (1980).

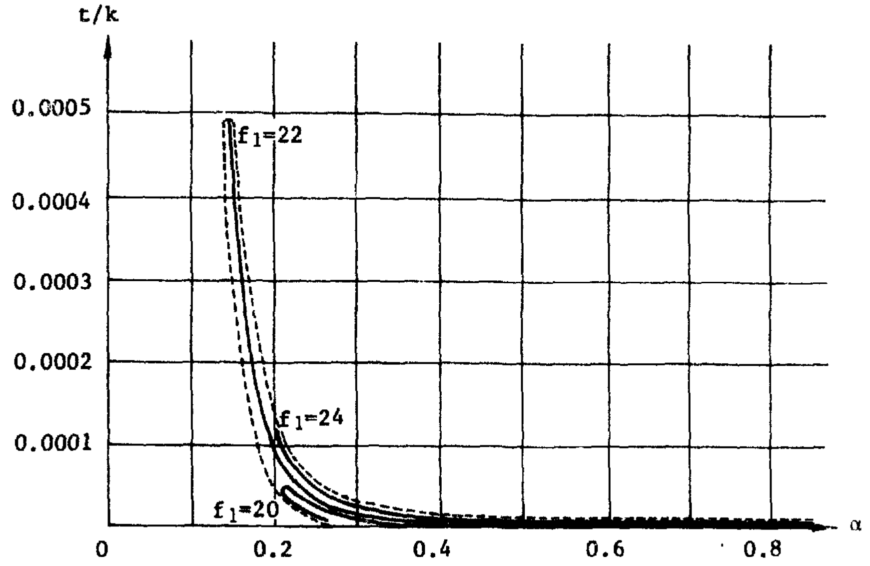

Fig. 13. Equilibrium condition on t/k for tricentric urban configuration of type B.

type A and type B are: (i) the range of α, where the equilibrium urban configurations exist extends only from 0.027 to 0.094 for type A, while it extends from 0.143 to infinity for type B, (ii) the value of t/k is fairly large (from 0.002 to 0.0155) for type A, but very small (from infinitesimally small to 0.00048) for type B, and (iii) the size of *BD* at the center overwhelms that of the subcenter for type A, whereas these sizes are approximately equal (the ratio ranges from 1.125 to 1.85) for type B. In consequence, the city can exhibit a tricentric urban configuration of type A when agglomeration economies are fairly large (i.e., α is small) and the commuting cost is fairly high (compared with the value of parameter k), and can exhibit that of type B when agglomeration economies are small (i.e., α is large) and the commuting rate is small.

When our city has more than three business districts, a variety of sizes of *BD*s, *RA*s and commuting patterns can be considered. However, it is extremely complicated and difficult, although not impossible, to analyze the existence and properties of those multicentric urban configurations with more than three centers, and the analyses of these configurations are left for the future.

Summarizing the analysis in this section, we conclude that our problem of equilibrium urban land use has several types of solutions depending on the values of parameters, which, in the following discussion, will be notationally represented as

u_0 = completely mixed urban configuration,
u_* = incompletely mixed urban configuration,

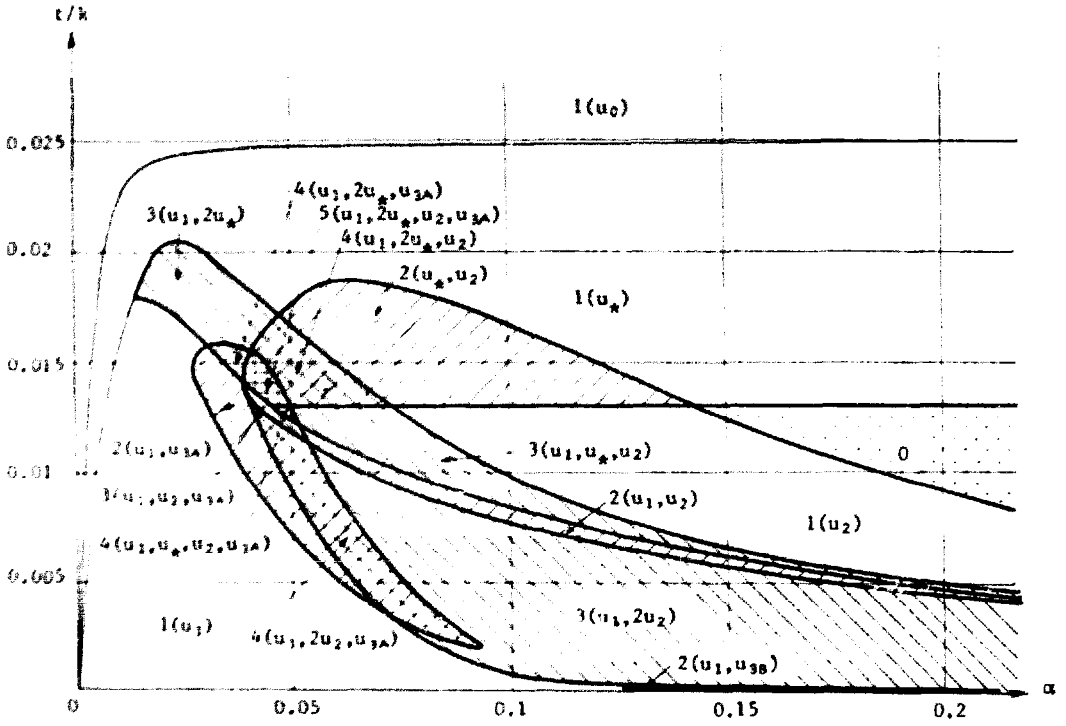

Fig. 14. Number of equilibrium solutions.

u_1 = monocentric urban configuration,
u_2 = duocentric urban configuration,
u_{3A} = tricentric urban configuration (type A),
u_{3B} = tricentric urban configuration (type B).

Another interesting result is that there are multiple equilibrium solutions under a wide range of parameter values. Fig. 14 summarizes the region and number of multiple equilibrium solutions. According to the figure, at maximum there exist five different equilibria, $\{u_1$, two u_* with different sizes of integrated districts, u_2, $u_{3A}\}$, under the same set of parameter values. In contrast, there is a region where no equilibrium solution exists. Of course, these results are tentative; further analyses might bring to light more than these five equilibria, as even new types of configurations.[10]

3.3. *Effects of other parameters*

In the previous numerical analysis, we fixed the values of parameters, S_h, S_b, L_b and N. Hence, the results in sections 3.2 and 3.3 are dependent upon the particular choice of numerical values for these parameters. A complete

[10] It is expected that incompletely mixed duocentric urban configuration will fill the dotted area (0 solution area) in fig. 14.

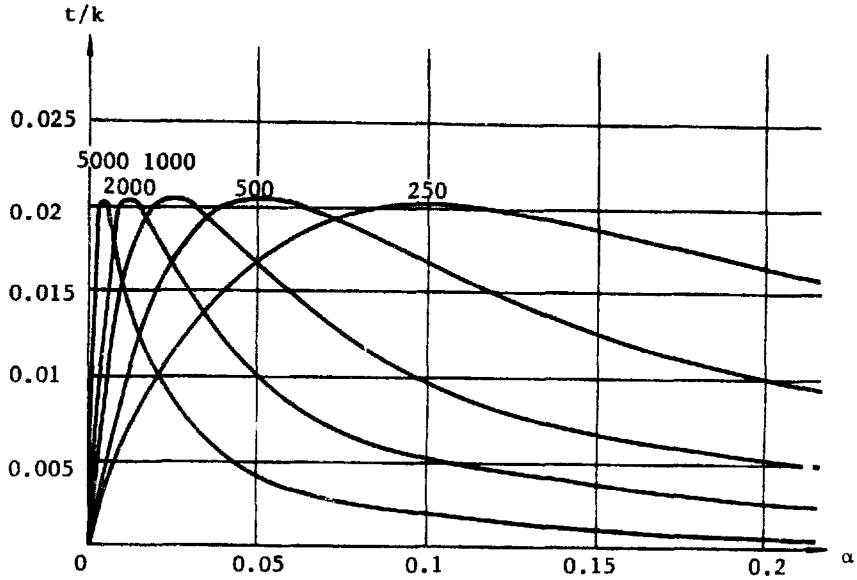

Fig. 15. Equilibrium condition on t/k for different population sizes in monocentric configuration.

discussion of the effects of these parameters is, unfortunately, beyond the scope of this paper. However, the effect of the total number of households, N, on the equilibrium urban configurations is clear; with a relatively simple analysis, we can show that.[11]

Property 1. Suppose we have an equilibrium urban configuration under a set of parameter values, $\{t, k, \alpha, N, S_h, S_b, L_b\}$, and suppose we change N to nN, where n is a positive constant. Then, that urban configuration remains an equilibrium configuration under a new set of parameter values, $\{t, k, \alpha', nN, S_h, S_b, L_b\}$, if and only if $\alpha' = \alpha/n$.

For example, fig. 15 illustrates the effect of population change on the equilibrium urban configuration in the case of a monocentric pattern. We see from this figure that the larger is N, the smaller is t/k at each α for the monocentric urban configuration to remain in equilibrium. This implies that, as population increases, the city is less likely to exhibit a monocentric pattern.

4. Total net land rent under each urban configuration

In this section, we calculate the total net land rent which corresponds to each urban configuration. Since the agricultural land rent is the opportunity cost (or rent forgone) for the development of the urban area, we deduct it

[11] For the proof of Property 1, see Ogawa and Fujita (1980).

from the total (gross) land rent, and obtain the total net land rent for each urban configuration. This calculation prepares us to carry out the analysis in the next section, and enables us to compare equilibrium urban configurations and optimum urban configurations.[12]

We employ the same set of values for parameters $\{N, S_h, S_b, L_b\} = \{1000, 0.1, 1, 10\}$ as in section 3, and, as before, show the total net land rent on $\{t/k, \alpha\}$-space. Consider an example of the monocentric urban configuration. From (3.15), the land rent function, $R(x)$, for the case of the monocentric urban configuration is given by

$$\begin{aligned} R(x) &= Ak + B_1 t + R_A \quad \text{for } x \in [0, f_1], \\ &= B_2 t + R_A \qquad \text{for } x \in [f_1, f_2], \end{aligned}$$

where $A = (1/S_b)\,(F(x) - F(f_1))$, $B_1 = -(L_b/S_b)\,(f_1 - x) + (1/S_h)\,(f_2 - f_1)$, and $B_2 = (1/S_h)\,(f_2 - x)$. Hence, the total net land rent (TNR) of the monocentric pattern is given by

$$\begin{aligned} TNR &= 2\int_0^{f_2} (R(x) - R_A)\,\mathrm{d}x \\ &= 2\left\{\int_0^{f_1} (Ak + B_1 t)\,\mathrm{d}x + \int_{f_1}^{f_2} B_2 t\,\mathrm{d}x\right\} \\ &= 2(A'k + B't), \end{aligned}$$

where $A' = \int_0^{f_1} A\,\mathrm{d}x$, $B' = \int_0^{f_1} B_1\,\mathrm{d}x + \int_{f_1}^{f_2} B_2\,\mathrm{d}x$. Or, if we divide both sides by k, we have

$$\frac{TNR}{k} = 2\left(A' + B'\frac{t}{k}\right), \tag{4.1}$$

where t/k is subject to the condition (3.14). The same procedure applies to other urban configurations.

Since the value of the production parameter k is exogenously given and is common for all urban configurations, it is convenient to compare the values of TNR/k on the $\{t/k, \alpha\}$-space. Fig. 16 shows the profiles of TNR/k-surface at various values of α. When $\alpha = 0.01$, the values of TNR/k changes continuously with t/k. However, when $\alpha = 0.02$, the TNR/k-surface has a folding part where the values of TNR/k for an incompletely mixed pattern

[12] For the comparison of equilibrium urban configurations and optimum urban configurations, see Ogawa and Fujita (1980).

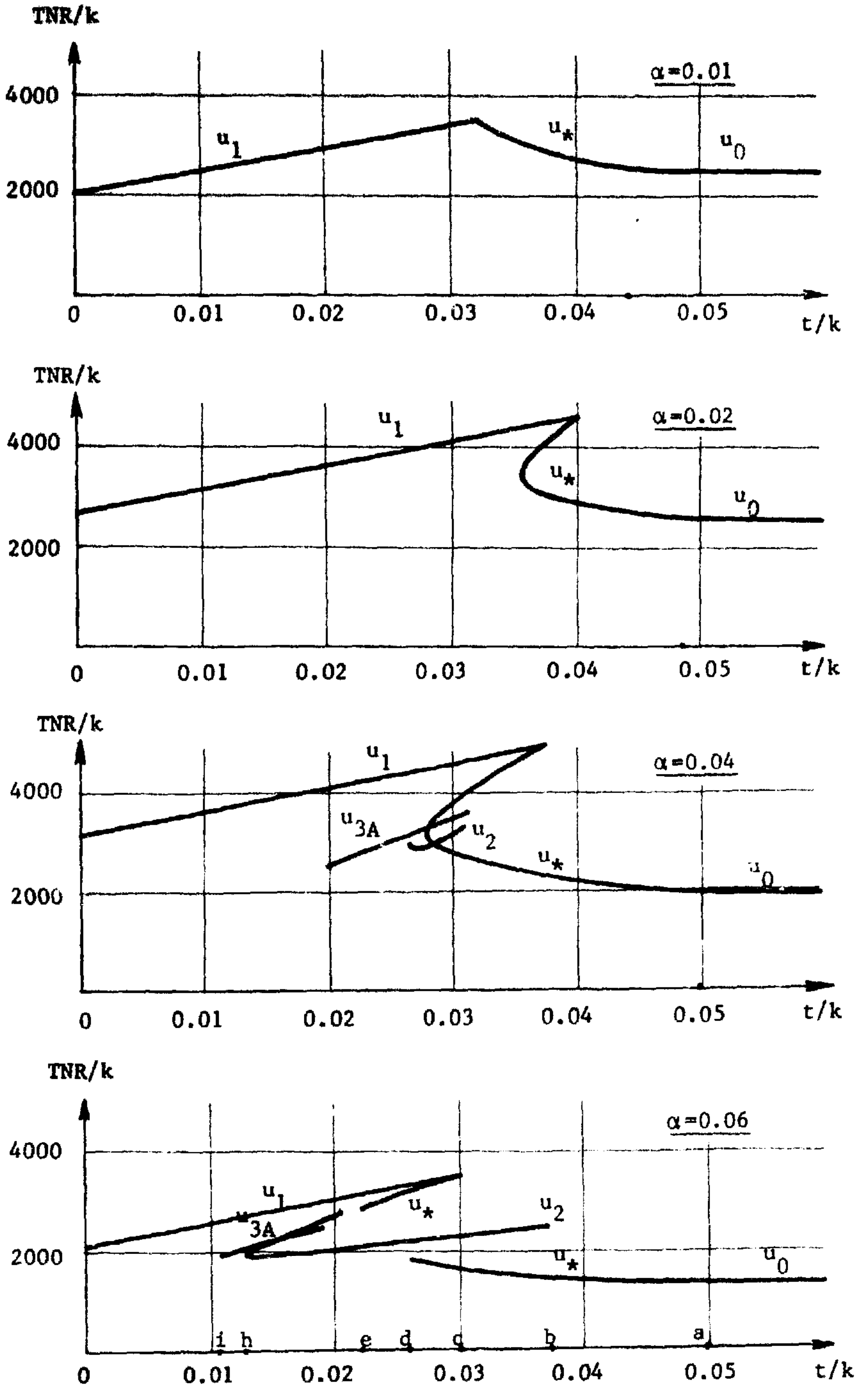

Fig. 16. The profiles of total net land rents.

(u_*) are dominated by those for the monocentric pattern (u_1). When α reaches a value around 0.03, a tricentric urban configuration of type A (u_{3A}) comes into the figure, and when α attains a value around 0.04, a duocentric urban configuration (u_2) emerges. As seen from the figures, there exists a region in $\{t/k, \alpha\}$-space when a duocentric pattern has the maximum value of TNR/k among all the equilibrium patterns under the same set of parameter values; and that region expands as the value of t/k increases. On the other hand, tricentric patterns are always dominated by monocentric patterns with respect to the value of TNR/k.

5. Structural transition of urban configuration

The previous analysis has shown that our city exhibits a variety of types of urban configurations depending on the values of the parameters. That is to say, a change in parameter values will cause a change in the spatial structure of the city; for example, from a monocentric pattern to a multicentric pattern. In this section, we examine this change in spatial structure which is due to change in parameter values. We call this change the *structural transition of the urban configuration.*[13] We first analyze the modes of structural transition on $\{t/k, \alpha\}$-space while fixing the values of the rest of parameters, as we did in sections 3 and 4, to $\{N, S_k, S_b, L_b\} = \{1000, 0.1, 1, 10\}$. Then, in the last part of this section, we briefly discuss the mode of structural transition which results from a change in the parameter, N.

As the first example of the modes of structural transition on $\{t/k, \alpha\}$-space, we consider the case where the value of t/k changes while the value of α is kept at 0.06. Suppose that the initial urban configuration consists of a completely mixed pattern u_0, and that the value of t/k continuously decreases (for the following discussion, refer to fig. 16 with $\alpha = 0.06$ and fig. 17). Then, as t/k decreases, the equilibrium urban configuration remains u_0 until t/k reaches the point, a, and then changes from u_0 to u_* when t/k crosses a. The incompletely mixed pattern persists as long as t/k reaches d. Notice that when t/k is on the path between b and d, there exist other urban configurations, that is, u_2 and u_1, both of which are equilibrium patterns. However, there is no particular reason for the city to change from u_* to u_2 at b, or from u_* to u_1 at c. Hence, it seems reasonable to assume that the incompletely mixed pattern persists until t/k reaches point d. But, when t/k

[13] Our analysis here is essentially a comparative static analysis of the equilibrium solution. Although the usefulness of the comparative static approach to the study of urban spatial structure is debatable, it does suggest, however, the direction of the change in spatial structure due to a change in parameter values. To carry out a more complete analysis of the structural transition of urban configurations, we must first specify the behavior of the system when it is out of equilibrium, then, we will be able to obtain the time-path of the actual change. But such dynamic analysis is outside the scope of this paper.

decreases beyond d, the same u^* cannot survive any longer; for the city to remain in equilibrium, it must take either u_1, u_2 or u_* with a smaller size of integrated district. In either case, the change in spatial structure is discontinuous: (i) If the spatial structure changes from u^* to u_1 at d, then an integrated district of, approximately, size 100 disappears suddenly. If this is the case, u_1 continues as long as t/k decreases. (ii) (iii) If the city changes its configuration from u_* to u_2 at d, then u_2 continues until t/k reaches h. Beyond h, a further decrease of t/k brings about a discontinuous structural transition from u_2 to either u_1 or u_{3A}. (iv) (v) (vi) If the spatial structure changes, at d, from u^* with an integrated district of size 100 to u^* with an integrated district of size 2, then u continues until e, changing thereafter to either u_1 or u_2. Schematically, these modes of structural transition can be represented by the following sequential changes in urban configuration.

(i) $u_0 \Rightarrow u_* \xrightarrow{d} u_1$,
(ii) $u_0 \Rightarrow u_* \xrightarrow{d} u_2 \xrightarrow{h} u_1$,
(iii) $u_0 \Rightarrow u_* \xrightarrow{d} u_2 \xrightarrow{h} u_{3A} \xrightarrow{i} u_1$,
(iv) $u_0 \Rightarrow u_* \xrightarrow{d} u_*$ (smaller ID) $\xrightarrow{e} u_1$,
(v) $u_0 \Rightarrow u_* \xrightarrow{d} u_*$ (smaller ID) $\xrightarrow{e} u_2 \xrightarrow{h} u_1$,
(vi) $u_0 \Rightarrow u_* \xrightarrow{d} u_*$ (smaller ID) $\xrightarrow{e} u_2 \xrightarrow{h} u_{3A} \xrightarrow{i} u_1$,

where '$\Rightarrow$' represents a smooth continuous change in spatial structure, while '$\xrightarrow{\cdot}$' represents a discontinuous change at '·'. These discontinuous changes in spatial structure may be called *catastrophic structural transitions*. As we have stated previously, without further assumptions on the behavior of the system, we cannot predict which urban configuration the city actually exhibits among the six possibilities.

On the other hand, if t/k continuously increases while keeping $\alpha = 0.06$, we can see from fig. 16 ($\alpha = 0.06$) that there are two possible modes of structural transition:

(i′) $u_1 \xrightarrow{c} u_* \Rightarrow u_0$,
(ii′) $u_1 \xrightarrow{c} u_2 \xrightarrow{b} u_* \Rightarrow u_0$.

Note that transitions (i′) and (ii′) are not exact reverse processes of (i) and (ii), respectively. For example, in (i), a catastrophic structural transition takes place at d; but in (i′), it place at c. That is, structural transitions are generally not reversible.

Next, as our second example, let us study the modes of structural transition when the value of α changes while the value of t/k is kept at 0.009. If α decreases the two following modes of structural transition are possible (refer to fig. 17):

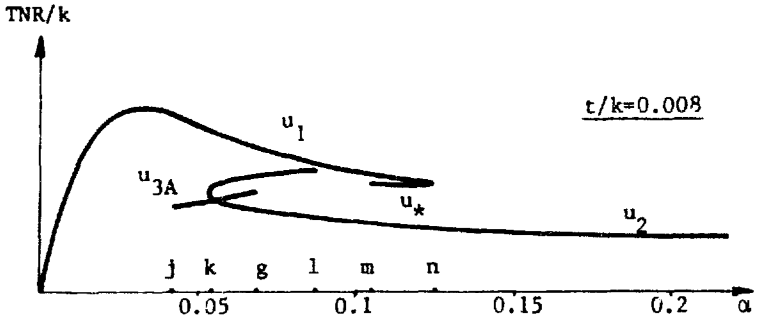

Fig. 17. Total net land rent curve for $t/k = 0.008$.

(i) $u_2 \xrightarrow{k} u_1$,
(ii) $u_2 \xrightarrow{k} u_{3A} \xrightarrow{j} u_1$.

On the other hand, if α increases there is only one mode of structural transition:

(i′) $u_1 \xrightarrow{n} u_2$.

Note again that transition (i′) is not the exact reverse process of (i).

We can perform similar analyses of modes of structural transition on other paths on $\{t/k, \alpha\}$-space. Finally, the spatial structure may change as the population, N, changes, and both t/k and α remain unchanged. We see, from Property 1 in section 3.4, that the effect of population change is the same as that of α. Namely, an increase (decrease) of N causes the same structural transition as an increase (decrease) of α as seen from fig. 15. This implies that population growth may reduce the relative advantage of concentration in the monocentric pattern compared to the case in which population remains small, and hence, the city is less likely to exhibit monocentricity. Such a structural change from a monocentric pattern to other non-monocentric or multicentric patterns caused by population increase may be catastrophic when the population exceeds some critical level.[14]

6. Conclusion

One of the main findings of the analysis in this paper is that the city may undergo a catastrophic structural transition when the parameters take critical

[14]In actual cities, it is difficult to observe catastrophic changes in urban spatial structure because of the durability of buildings and imperfect substitutability between residential and non-residential buildings.

values. This phenomenon of catastrophic change in the urban configuration has scarcely been examined in current urban land use theory.

For example, recent urban land use models conclude that decentralization will certainly result from increases in income and decreases in the time and cost of commuting. But after decentralization, urban spatial structure remains essentially the same: it consists of the CBD and some concentric rings of residential area with different incomes and classes. This is also true in the case of population change. However, as we have shown, the monocentric urban configuration, the preferred paradigm in current urban land use models, may not persist at equilibrium when the city's population and commuting rate change. Accordingly, some conclusions obtained in those models which assume monocentricity are brought into serious question and should be re-examined.

Once a catastrophic structural transition of the urban configuration has been recognized, the ensuing problem is to understand the underlying dynamics which generate it. In this paper, the analysis of structural transition was carried out in the comparative static sense; but the notion of structural transition, i.e., a change of state, is dynamic in nature. Every parameter changes with time, and so does the urban configuration. Therefore, it may be useful to apply Thom-type catastrophe theory to this problem in order to understand better the process of structural transition. It is, however, beyond the s e of this study and must await treatment in some future work.

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