CHAPTER 2

Locational choice of the household

2.1 Introduction

Any household that moves to a city and has to choose a residence is faced with a complex set of decisions. We can view this situation as a trade-off problem, in which there are three basic factors: accessibility, space, and environmental amenities.

Accessibility includes both pecuniary and time costs associated with getting to and from work, visiting relatives and friends, shopping, and other such activities. The space factor consists of the need for some land as well as the size and quality of the house itself. Finally, environmental amenities include natural features such as hills and scenic views as well as neighborhood characteristics ranging from quality of schools and safety to racial composition.

In making a residential choice a household must weigh all three factors appropriately, yet also meet budget and time constraints. For example, a location with good accessibility usually commands a high price for space. So the household may have to sacrifice space for accessibility. Accessible locations, however, are typically lacking in environmental quality. Thus, the household also confronts a choice between accessibility and environment.

Even though in actual practice all three factors are important for making a residential choice, when constructing theory it is difficult to treat all factors at once. Following the time-honored wisdom of theory building, we shall begin by studying a pure case and expand the framework later on. Part I examines the trade-off between accessibility and space in residential choice. Part II introduces environmental factors.

2.2 Basic model of residential choice

The development of our understanding of residential land use begins with the basic model, which focuses on the trade-off between accessibility and space. The model rests on a set of assumptions about the spatial character of the urban area:

1. The city is monocentric; that is, it has a single prespecified center of fixed size called the central business district (CBD). All job opportunities are located in the CBD.
2. There is a dense, radial transport system. It is free of congestion. Furthermore, the only travel is that of workers commuting between residences and work places. (Travel within the CBD is ignored.)
3. The land is a featureless plain. All land parcels are identical and ready for residential use. No local public goods or bads are in evidence, nor are there any neighborhood externalities.

In this context, the only spatial characteristic of each location in the city that matters to households is the distance from the CBD. Thus, the urban space can be treated as if it were one-dimensional.

Consider a household that seeks a residence in the city. As is typical in the economic analysis of consumer behavior, we assume that the household will maximize its utility subject to a budget constraint.[1] We specify the utility function $U(z, s)$, where z represents the amount of *composite consumer good,* which includes all consumer goods except land, and s the consumption of land, or the *lot size* of the house.[2] The composite consumer good is chosen as the numeraire, so its price is unity. The household earns a fixed income Y per unit time, which is spent on the composite good, land, and transportation. If the household is located at distance r from the CBD, the budget constraint is given by $z + R(r)s = Y - T(r)$, where $R(r)$ is the rent per unit of land at r, $T(r)$ is the transport cost at r, and hence $Y - T(r)$ is the *net income* at r. So we can express the residential choice of the household as

$$\max_{r,z,s} U(z, s), \qquad \text{subject to} \quad z + R(r)s = Y - T(r), \tag{2.1}$$

where $r \geq 0$, $z > 0$, $s > 0$. This is called the *basic model* of residential choice.

By definition the choice of r is restricted to the range $r \geq 0$. It is reasonable to assume that the subsistence of the household needs some positive amounts of both z and s. That is, both goods are *essential*. Therefore, we require the utility function $U(z, s)$ to be defined only for positive z and s. This is equivalent to saying that indifference curves in the consumption space do not cut axes. Considering this, we introduce the following set of assumptions:[3]

Assumption 2.1 (well-behaved utility function). The utility function is continuous and increasing at all $z > 0$ and $s > 0$; all

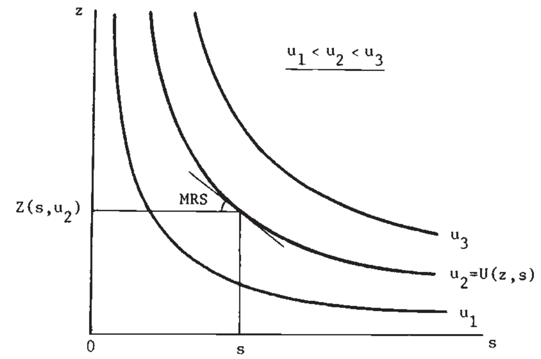

Figure 2.1. The consumption space and indifference curves.

indifference curves are strictly convex and smooth, and do not cut axes.

Assumption 2.2 (increasing transport cost). Transport cost $T(r)$ is continuous and increasing at all $r \geq 0$, where $0 \leq T(0) < Y$ and $T(\infty) = \infty$.

These assumptions are always taken to hold in the subsequent analysis. On the basis of Assumption 2.1, indifference curves in the consumption space can be depicted as in Figure 2.1. Recall that an indifference curve is the locus of all consumption bundles from which the household derives the same utility level. The indifference curve with utility level u can be expressed in implicit form as $u = U(z, s)$. Or solving $u = U(z, s)$ for z, the *equation of the indifference curve* with utility level u can be stated as

$$z = Z(s, u). \tag{2.2}$$

By definition, $Z(s, u)$ represents the amount of composite good that is necessary to achieve utility level u when the lot size of the house is s (see Figure 2.1).

Throughout our study, differential calculus will often be used to make the analysis simple. Whenever differential calculi are involved, we are also implicitly assuming that utility function $U(z, s)$ is twice continuously differentiable in z and s (i.e., all its second-order partial derivates exist and are continuous), and transport cost function $T(r)$ is continuously differentiable in r. Then in terms of differential calculus, the fact that utility function is increasing in z and s (Assumption 2.1) means[4]

$$\frac{\partial U(z, s)}{\partial z} > 0, \qquad \frac{\partial U(z, s)}{\partial s} > 0. \tag{2.3}$$

That is, the *marginal utility* of each good is positive. Note that this condition can be equivalently expressed as

$$\frac{\partial Z(s, u)}{\partial u} > 0, \qquad \frac{\partial Z(s, u)}{\partial s} < 0. \tag{2.4}$$

The term $-\partial Z(s, u)/\partial s$ is called the *marginal rate of substitution* (MRS) between z and s (Figure 2.1). Then the strict convexity of each indifference curve means that the MRS is diminishing in s:

$$-\frac{\partial^2 Z(s, u)}{\partial s^2} < 0. \tag{2.5}$$

Likewise, the fact that the transport cost function is increasing in r means

$$T'(r) > 0, \tag{2.6}$$

where $T'(r) \equiv dT(r)/dr$.

By directly solving the optimization problem implied by the basic model (2.1), we could ascertain the household's residential decision in a straightforward manner. But there is another approach, conceptually much richer, that leads to a desirable elaboration of theory. This approach, which mimics the von Thünen model of agricultural land use, requires the introduction of a concept called bid rent.

2.3 Bid rent function of the household

Bid rent is a conceptual device that describes a particular household's ability to pay for land under a fixed utility level. It is not to be confused with the market rent structure of the city, which arises from the interaction of many households. We define bid rent as follows:

Definition 2.1. The bid rent $\Psi(r, u)$ is the maximum rent per unit of land that the household can pay for residing at distance r while enjoying a fixed utility level u.

In the context of the basic model (2.1), bid rent can be mathematically expressed as

$$\Psi(r, u) = \max_{z,s} \left\{ \frac{Y - T(r) - z}{s} \;\middle|\; U(z, s) = u \right\}. \tag{2.7}$$

That is, for the household residing at distance r and selecting consumption bundle (z, s), $Y - T(r) - z$ is the money available for rent, or land payment, and $(Y - T(r) - z)/s$ represents the rent per unit of land at r.

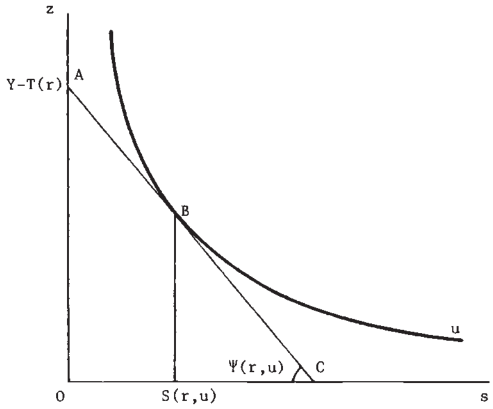

Figure 2.2. Bid rent $\Psi(r, u)$ and bid-max lot size $S(r, u)$.

According to Definition 2.1, therefore, bid rent $\Psi(r, u)$ is obtained when $(Y - T(r) - z)/s$ is maximized by the appropriate choice of a consumption bundle (z, s) subject to the utility constraint $U(z, s) = u$. Alternatively, in the maximization problem of (2.7), we may first solve the utility constraint $U(z, s) = u$ for z and obtain the equation of an indifference curve as (2.2). Then the bid rent function can be redefined as

$$\Psi(r, u) = \max_{s} \frac{Y - T(r) - Z(s, u)}{s}, \tag{2.8}$$

which is an unconstrained maximization problem.[5] When we solve the maximization problem of (2.7) or (2.8), we obtain the optimal lot size $S(r, u)$, which is called the *bid-max lot size.*[6]

Graphically, as depicted in Figure 2.2, *bid rent $\Psi(r, u)$ is given by the slope of the budget line at distance r that is just tangent to indifference curve u.*[7] To see this, let us generally denote the land rent at r by parameter R. Then the household's budget constraint at r can be generally expressed as $z + Rs = Y - T(r)$, or

$$z = (Y - T(r)) - Rs. \tag{2.9}$$

In Figure 2.2, under each value of land rent R, equation (2.9) defines a straight line that originates from point A and has the (absolute) slope R. If land rent R is greater than the slope of line AC, the budget line is entirely below the indifference curve u. This implies that in order to achieve the required utility level u, the household cannot pay land rent as high as R. Conversely, if land rent R is smaller than the slope of line AC, the budget line intersects indifference curve u. This implies that even under a slightly higher land rent, the household can achieve utility level u. Thus, we can conclude that bid rent $\Psi(r, u)$, that is, the highest land rent at r under which the household can achieve utility level u, is given by the slope of budget line AC. The tangency point B determines bid-max lot size $S(r, u)$. This graphical approach is useful to cement the definitions not only here, but also in our subsequent analysis. Next, notice that in the maximization problem of (2.8), function $(Y - T(r) - Z(s, u))/s$ is maximized in s at the point where the marginal change of the function with respect to s is zero. This leads to the next relation:[8]

$$-\frac{\partial Z(s, u)}{\partial s} = \frac{Y - T(r) - Z(s, u)}{s}. \qquad (2.10)$$

Solving this equation for s, we obtain the bid-max lot size $S(r, u)$.[9] Or, since at the optimal choice of s the right side of (2.10) equals $\Psi(r, u)$, condition (2.10) can be restated as

$$-\frac{\partial Z(s, u)}{\partial s} = \Psi(r, u). \qquad (2.11)$$

In terms of Figure 2.2, this means that at the tangency point B, the slope $-\partial Z(s, u)/\partial s$ ($\equiv$ MRS) of indifference curve u equals the slope $\Psi(r, u)$ of budget line AC.

Example 2.1. Suppose that the utility function in model (2.1) is given by the following *log-linear function:*

$$U(z, s) = \alpha \log z + \beta \log s, \qquad (2.12)$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta = 1$. It is not difficult to confirm that this utility function satisfies all the conditions of Assumption 2.1. The equation of the indifference curve is given as $Z(s, u) = s^{-\beta/\alpha} e^{u/\alpha}$. Solving the maximization problem of (2.8) by using condition (2.10), we have[10]

$$\Psi(r, u) = \alpha^{\alpha/\beta} \beta (Y - T(r))^{1/\beta} e^{-u/\beta}, \qquad (2.13)$$

$$S(r, u) = \beta(Y - T(r))/\Psi(r, u) = \alpha^{-\alpha/\beta}(Y - T(r))^{-\alpha/\beta} e^{u/\beta}. \qquad (2.14)$$

Now that we have introduced bid rent $\Psi(r, u)$ and bid-max lot size $S(r, u)$, which are concepts unique to land use theory,[11] it is helpful to relate them to familiar microeconomic notions. In this way, we will then be able to take advantage of the well-established tools of traditional economic analysis. To this end, let us return to Figure 2.2. This figure can be interpreted in several revealing ways.

To begin with, consider the following *utility-maximization problem* under land rent R and net income I:

$$\max_{z,s} U(z, s), \qquad \text{subject to} \quad z + Rs = I. \tag{2.15}$$

When we solve this problem, we obtain the optimal lot size,

$$\hat{s}(R, I), \tag{2.16}$$

as a function of R and I, which is called the *Marshallian (ordinary) demand function* for land. The maximum value of this problem is represented as

$$V(R, I) = \max_{z,s} \{U(z, s) | z + Rs = I\}, \tag{2.17}$$

which is called the *indirect utility function*. This gives the maximum utility attainable from net income I under land rent R. If we set $R = \Psi(r, u)$ and $I = Y - T(r)$, problem (2.15) becomes

$$\max_{z,s} U(z, s), \qquad \text{subject to} \quad z + \Psi(r, u)s = Y - T(r). \tag{2.18}$$

Now, we can interpret Figure 2.2 as indifference curve u being tangent to budget line AC from above at point B. Since the equation of line AC is $z + \Psi(r, u)s = Y - T(r)$, this means exactly that point B is the solution of problem (2.18), and u is its maximum value. Hence, setting $R = \Psi(r, u)$ and $I = Y - T(r)$ in (2.16) and (2.17), it must hold identically that

$$S(r, u) \equiv \hat{s}(\Psi(r, u), Y - T(r)), \tag{2.19}$$

$$u \equiv V(\Psi(r, u), Y - T(r)). \tag{2.20}$$

In other words, the maximum utility under land rent $\Psi(r, u)$ and net income $Y - T(r)$ equals u, and the bid-max lot size at utility u equals the Marshallian demand for land under land rent $\Psi(r, u)$.

Next, consider the following *expenditure-minimization problem* under land rent R and utility level u:

$$\min_{z,s} z + Rs, \qquad \text{subject to} \quad U(z, s) = u. \tag{2.21}$$

When we solve this problem, we obtain the optimal lot size,

$$\tilde{s}(R, u), \tag{2.22}$$

as a function of R and u, which is called the *Hicksian (compensated) demand function* for land. The minimum value of this problem is denoted by $E(R, u)$, that is,

Table 2.1. *Bid rent and related functions*

	Alonso	Solow	Marshall	Hicks
Bid rent	$\Psi(r, u)$	$\psi(I, u)$	—	—
Lot size (land)	$S(r, u)$	$s(I, u)$	$\hat{s}(R, I)$	$\tilde{s}(R, u)$
Indirect utility	—	—	$V(R, I)$	—
Expenditure	—	—	—	$E(R, u)$

$$E(R, u) = \min_{z,s}\{z + Rs \mid U(z, s) = u\}, \tag{2.23}$$

which is called the *expenditure function*. If we set $R = \Psi(r, u)$, problem (2.21) becomes

$$\min_{z,s}\ z + \Psi(r, u)s, \qquad \text{subject to} \quad U(z, s) = u. \tag{2.24}$$

Now, this time we can interpret Figure 2.2 as budget line AC being tangent to indifference curve u from below at B. Since the equation of line AC is $Y - T(r) = z + \Psi(r, u)s$, this means exactly that point B is the solution of problem (2.24), and $Y - T(r)$ is its minimum value.[12] Hence, setting $R = \Psi(r, u)$ in (2.22) and (2.23), it must hold identically that

$$S(r, u) \equiv \tilde{s}(\Psi(r, u), u), \tag{2.25}$$

$$Y - T(r) \equiv E(\Psi(r, u), u). \tag{2.26}$$

In other words, the minimum expenditure needed to reach utility u at land rent $\Psi(r, u)$ is $Y - T(r)$, and the bid-max lot size at utility u is identical to the Hicksian demand for land at utility u under land rent $\Psi(r, u)$.

Since the characteristics of indirect utility functions, expenditure functions, and Marshallian and Hicksian demands are all well known, identities (2.19), (2.20), (2.25), and (2.26) provide us with powerful tools that will be useful in the sequel.[13] Table 2.1 summarizes various functions introduced. [Functions $\psi(I, u)$ and $s(I, u)$ are to be introduced in Section 3.2.]

Next, we examine important properties of bid rent and bid-max lot size functions. Consider first how bid rent and bid-max lot size change with r. To this end, let the utility level be fixed at u, and take two distances such that $r_1 < r_2$. Then since $T(r_1) < T(r_2)$, we have $Y - T(r_1) > Y - T(r_2)$. Recall that bid rent $\Psi(r, u)$ at distance r is given by the slope of the budget line at r, which is just tangent to indifference curve u. Then from Figure 2.3 it is easily grasped that $\Psi(r_1, u) > \Psi(r_2, u)$ and $S(r_1, u) < S(r_2, u)$. That is, bid rent decreases in r and bid-max lot size increases

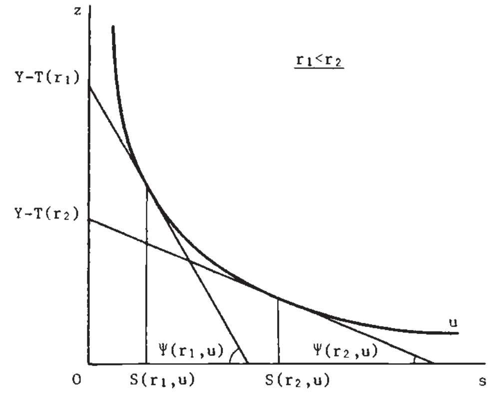

Figure 2.3. Changes in $\Psi(r, u)$ and $S(r, u)$ with an increase in r.

in r. These properties make intuitive sense. Given a reduction in net income, a household can retain its prior utility level only if the rental price of land is also reduced, enabling the household to substitute land for the composite good (the price of which is fixed at unity).

The rate of change of bid rent with respect to r can be calculated through an application of the *envelope theorem* to equation (2.8) as follows:[14]

$$\frac{\partial \Psi(r, u)}{\partial r} = -\frac{T'(r)}{S(r, u)} < 0. \tag{2.27}$$

Observe from equation (2.8) that an increase in r produces two effects on $\Psi(r, u)$. One is the *direct effect* that occurs through an increase in transport costs. A unit increase in commuting distance increases transport cost by the increment $T'(r)$, which in turn reduces the net income by the same amount; thus, the land payment ability per unit of land (i.e., the bid rent) decreases $T'(r)/S(r, u)$. The other effect is induced via changes in the optimal consumption bundle $(Z(S(r, u), u), S(r, u))$ as r increases. However, the envelope theorem indicates that this *induced effect* is negligible when changes in r are small. We are left then with only the direct effect shown above.

Combining the result of (2.27) and the identity $S(r, u) = \hat{s}(\Psi(r, u), u)$, we can calculate the rate of change of bid-max lot size with respect to r as

$$\frac{\partial S(r, u)}{\partial r} = \frac{\partial \hat{s}}{\partial R}\frac{\partial \Psi(r, u)}{\partial r} = -\frac{\partial \hat{s}}{\partial R}\frac{T'(r)}{S(r, u)} > 0, \tag{2.28}$$

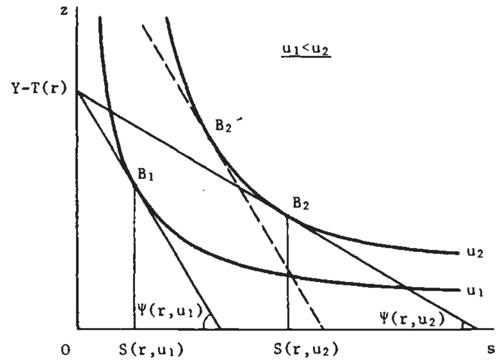

Figure 2.4. Changes in $\Psi(r, u)$ and $S(r, u)$ with respect to u.

which is positive, since its own price effect on Hicksian demand $\tilde{s}$ is always negative (see Appendix A.3).

Next, let us investigate how bid rent and bid-max lot size change with utility level. Let distance r be fixed, and choose two utility levels such that $u_1 < u_2$. Then since indifference curve u_2 lies above curve u_1, it is easy to see from Figure 2.4 that $\Psi(r, u_1) > \Psi(r, u_2)$. This conclusion also makes intuitive sense in that a household can attain higher utility with fixed net income only if land rent is reduced. The impact of a utility change on the bid-max lot size is more complex. According to Figure 2.4, an increasing utility level causes an increase in the bid-max lot size. This result, however, cannot always hold true without some additional assumptions. The following assumption represents a sufficient condition for ensuring such a result:

> ***Assumption 2.3 (normality of land).*** The income effect on the Marshallian demand for land is positive.

To explain the meaning of this assumption, it is convenient to consider the movement in Figure 2.4 from point B_1 [the original consumption bundle under the land rent $\Psi(r, u_1)$] to point B_2 [the new consumption bundle under a lower land rent $\Psi(r, u_2)$] as the sum of the movement from B_1 to B_2' and that from B_2' to B_2. Here B_2' represents the consumption bundle that will be achieved when the land rent is fixed at $\Psi(r, u_1)$ and the income increases from $Y - T(r)$ to the one associated with the dashed budget line. The normality of land means that the movement from B_1 to B_2' (i.e., the *income effect*) causes an increase in land consumption. Then since the movement from B_2' to B_2 [i.e., the *substitution effect* associated with a reduction in land rent from $\Psi(r, u_1)$ to $\Psi(r, u_2)$ while the utility level is held constant at u_2] always causes an increase in land consumption, we necessarily have that $S(r, u_1) < S(r, u_2)$. Since the normality of land is empirically supported, we assume that Assumption 2.3 also always holds in the subsequent analysis. Notice that the normality of land means graphically that at each fixed s, slopes of indifference curves (in absolute value) become greater as u increases. Notice also that the log-linear utility function of Example 2.1 satisfies this assumption.

We can calculate the rate of change in bid rent with respect to u by applying the envelope theorem to equation (2.8) as follows,

$$\frac{\partial \Psi(r, u)}{\partial u} = -\frac{1}{S(r, u)} \frac{\partial Z(s, u)}{\partial u} < 0, \tag{2.29}$$

which is negative since $\partial Z/\partial u > 0$ from (2.4). So recalling identity $S(r, u) \equiv \hat{s}(\Psi(r, u), Y - T(r))$, we have

$$\frac{\partial S(r, u)}{\partial u} = \frac{\partial \hat{s}}{\partial R} \frac{\partial \Psi(r, u)}{\partial u} > 0. \tag{2.30}$$

The positivity is obtained since $\partial \Psi/\partial u < 0$ and since the normality of land implies that its own price effect $\partial \hat{s}/\partial R$ on Marshallian demand $\hat{s}$ is negative (see Appendix A.3).

Finally, the continuity of transport cost function and the assumption of a well-behaved utility function imply that both the bid rent and bid-max lot size functions are continuous in r and u. Therefore, summarizing the discussion above, we can conclude as follows:

Property 2.1

(i) Bid rent $\Psi(r, u)$ is continuous and decreasing in both r and u (decreasing until Ψ becomes zero).

(ii) Bid-max lot size $S(r, u)$ is continuous and increasing in both r and u (increasing until S becomes infinite).

From (i) the general shape of bid rent curves can be depicted as in Figure 2.5; from (ii) above, the general shape of *(bid-max) lot size curves* can be depicted as in Figure 2.6. Each bid rent curve (lot size curve) is downward- (upward-) sloped. With an increase in utility level, bid rent curves (lot size curves) shift downward (upward). Each lot size curve approaches infinity at the distance where the corresponding bid rent curve intersects the r axis.[15]

Bid rent curves need not always be convex as depicted in Figure 2.5. But they are if we assume that the transport cost function is linear or

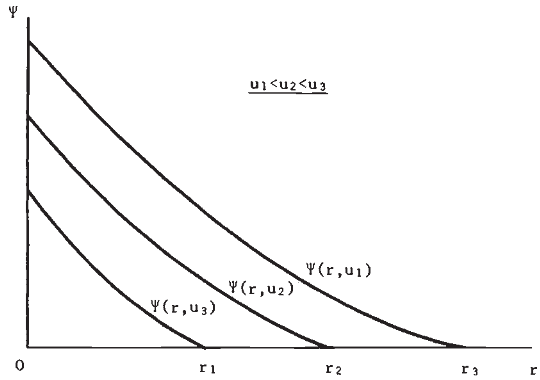

Figure 2.5. General shapes of bid rent curves.

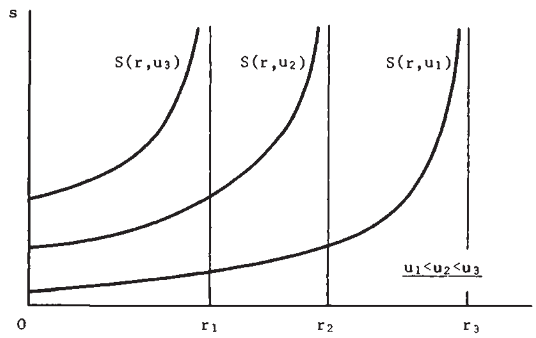

Figure 2.6. General shapes of lot size curves.

concave in distance r so that $T''(r) \equiv d^2T(r)/dr^2 \leq 0$. From (2.27),

$$\frac{\partial^2\Psi(r, u)}{\partial r^2} = -\frac{T''(r)}{S(r, u)} + \frac{T'(r)}{S(r, u)^2}\frac{\partial S(r, u)}{\partial r}. \tag{2.31}$$

$T'(r) > 0$ by assumption, and $\partial S(r, u)/\partial r > 0$ from (2.28). Hence, if $T''(r) \leq 0$, then $\partial^2\Psi(r, u)/\partial r^2 > 0$, which means that bid rent curves are strictly convex. A linear or concave transport cost function is one in which the marginal transport cost is nonincreasing; this is the most commonly observed case.

Property 2.2. If the transport cost function is linear or concave in distance, then bid rent curves are strictly convex.

Next, recall the following well-known characteristics of the indirect utility function (see Appendix A.3):

Property 2.3

(i) $V(R, I)$ is continuous at all $R > 0$ and $I > 0$.
(ii) $V(R, I)$ is decreasing in R and increasing in I.

Under the differentiability assumption of utility function, (ii) means

$$\frac{\partial V(R, I)}{\partial R} < 0, \qquad \frac{\partial V(R, I)}{\partial I} > 0. \tag{2.32}$$

If $R(r) = \Psi(r, u)$, then, of course, $V(R(r), Y - T(r)) = V(\Psi(r, u), Y - T(r))$. Since V is decreasing in R, we can also conclude that $V(R(r), Y - T(r))$ is greater (smaller) than $V(\Psi(r, u), Y - T(r))$ as $R(r)$ is smaller (greater) than $\Psi(r, u)$:

Property 2.4. At each r,

$$V(R(r), Y - T(r)) \gtreqless V(\Psi(r, u), Y - T(r)) \quad \text{as} \quad R(r) \lesseqgtr \Psi(r, u).$$

This property also turns out to be very useful in the subsequent analysis.

In closing this section, we make the observation that bid rent curves are indifference curves defined in urban space (consisting of the dimensions of distance and rent). Identity (2.20) implies that if the actual land rent curve $R(r)$ of the city coincided everywhere with a bid rent curve $\Psi(r, u)$, the household could obtain the same maximum utility u at every location by appropriately choosing its consumption bundle. Thus, the household would be indifferent between alternative locations. Since for each indifference curve in Figure 2.1 there exists a bid rent curve in Figure 2.5, the bid rent function can be thought of as a transformation that maps the indifference curves in commodity space into corresponding curves in urban space. With these indifference curves defined in urban space, we will be able to analyze graphically the locational choice of the household. Moreover, since bid rent curves are stated as a pecuniary bid per unit of land, they are comparable among different land users. We will therefore be able to analyze competition for land among different agents, again graphically in urban space.

2.4 Equilibrium location of the household

We are now ready to examine how the equilibrium location of the household is determined under a given land rent configuration of the city.[16]

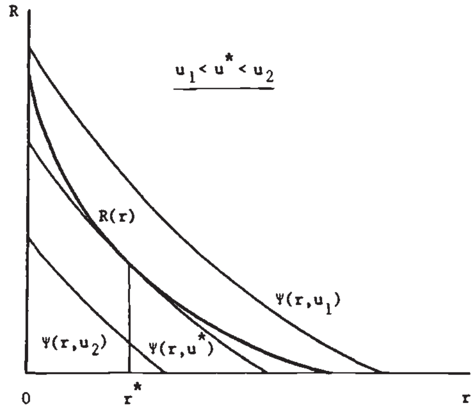

Figure 2.7. Determination of the equilibrium location.

The market land rent curve is given by $R(r)$, and the household takes it as an exogenous factor. The residential choice behavior of the household is represented by the basic model of (2.1).

We can approach the equilibrium location problem graphically, as shown in Figure 2.7. Here a set of bid rent curves is superimposed on the market rent curve $R(r)$. By inspection, it is evident from the figure that the equilibrium location of the household is distance r^* at which a bid rent curve $\Psi(r, u^*)$ is tangent to the market rent curve $R(r)$ from below. That is, when the household decides to locate somewhere in the city, it is obliged to pay the market land rent. At the same time, the household will maximize its utility. Since the utility of bid rent curves increases toward the origin, the highest utility will be achieved at a location at which a bid rent curve is tangent to the market rent curve from below. This result can be stated informally as the following rule:

> ***Rule 2.1'***. The equilibrium location of the household is that location at which a bid rent curve is tangent to the market rent curve from below.[17]

This rule can be restated in terms of the indirect utility function of (2.17). Let us call the maximum utility that the household can achieve in the city the *equilibrium utility of the household,* denoted by u^*. Recall that given the market rent curve $R(r)$, $V(R(r), Y - T(r))$ gives the maximum utility attainable for the household at each location r. Hence, u^* is the equilibrium utility of the household, and r^* is an optimal location if and only if

$$u^* = V(R(r^*), Y - T(r^*)) \tag{2.33}$$

and

$$u^* \geq V(R(r), Y - T(r)) \quad \text{for all} \quad r. \tag{2.34}$$

From Property 2.4, these conditions can be restated as

$$R(r^*) = \Psi(r^*, u^*)$$

and

$$R(r) \geq \Psi(r, u^*) \quad \text{for all} \quad r.$$

Therefore, Rule 2.1′ can be formally restated as follows:

> ***Rule 2.1 (individual location equilibrium).*** Given the market rent curve $R(r)$, u^* is the equilibrium utility of the household, and r^* is an optimal location if and only if
>
> $$R(r^*) = \Psi(r^*, u^*) \quad \text{and} \quad R(r) \geq \Psi(r, u^*) \quad \text{for all} \quad r. \tag{2.35}$$

Note that this rule is valid under any shape of curves $R(r)$ and $\Psi(r, u)$. At this point, we designate the bid rent curve $\Psi(r, u^*)$ that corresponds to the equilibrium utility u^* as the *equilibrium bid rent curve*.

Given that curves $R(r)$ and $\Psi(r, u^*)$ are smooth at r^*, the fact that two curves are tangent at r^* implies

$$\frac{\partial \Psi(r^*, u^*)}{\partial r} = R'(r^*), \tag{2.36}$$

where $R'(r) \equiv dR(r)/dr$. Thus, recalling equation (2.27), we have

$$T'(r^*) = -R'(r^*)S(r^*, u^*). \tag{2.37}$$

This result, called *Muth's condition,* asserts that at the equilibrium location the marginal transport cost $T'(r^*)$ equals the marginal land cost saving, $-R'(r^*)S(r, u^*)$. If this were not the case, the household could achieve greater utility by moving [closer to the CBD if $T'(r^*) > -R'(r^*)S(r^*, u)$; farther from the CBD if $T'(r^*) < -R'(r^*)S(r^*, u)$].

The *equilibrium lot size* at optimal location r^* is, by definition, the Marshallian demand for land, $\hat{s}(R(r^*), Y - T(r^*))$. From (2.35) and identity (2.19), this in turn equals the bid-max lot size $S(r^*, u^*)$:

$$\hat{s}(R(r^*), Y - T(r^*)) = S(r^*, u^*). \tag{2.38}$$

Example 2.2. In the context of the log-linear utility function of Example 2.1, let us suppose further that

$$R(r) = Ae^{-br}, \qquad T(r) = ar,$$

where A, a, and b are all positive constants. Then recalling (2.13) and (2.14), and using conditions (2.35) and (2.37), we can obtain the equilibrium location (i.e., optimal location) r^* of the household as follows:

$$r^* = \frac{Y}{a} - \frac{1}{b\beta},$$

provided that it is positive; otherwise, $r^* = 0$.

Thus far we have examined only the locational decision of a single household. We can now extend the analysis and ask what land use pattern will arise given many different households having different bid rent functions.

Suppose there are two households, i and j, having bid rent functions $\Psi_i(r, u)$ and $\Psi_j(r, u)$, respectively.[18] A general rule for ordering equilibrium locations of different households with respect to the distance from the CBD is as follows:

> ***Rule 2.2.*** If the equilibrium bid rent curve $\Psi_i(r, u_i^*)$ of household i and the equilibrium bid rent curve $\Psi_j(r, u_j^*)$ of household j intersect only once and if $\Psi_i(r, u_i^*)$ is steeper than $\Psi_j(r, u_j^*)$ at the intersection, then the equilibrium location of household i is closer to the CBD than that of household j.

In short, *a steeper equilibrium bid rent curve corresponds to an equilibrium location closer to the CBD*. This result is depicted in Figure 2.8. Note that neither household's equilibrium bid rent curve can dominate the other's over the whole urban space. If this were so, Rule 2.1, which states that each equilibrium bid rent curve must be tangent to $R(r)$ from below, would be violated. But if one curve cannot entirely dominate the other, then both curves must intersect at least once. In Figure 2.8, this occurs at distance x. Since the curve for household i is represented here as the steeper one, the equilibrium bid rent curve of household i dominates that of household j to the left of x. The reverse is true to the right of x. Hence, the equilibrium location r_i^* (r_j^*) of household i (j) must be to the left (right) of x.[19]

In order to apply the rule just stated, we must know beforehand which equilibrium bid rent curve is steeper at the intersection. In general, this information is difficult to obtain *a priori*. Matters can be greatly simplified, however, if we are able to determine the *relative steepness of bid rent functions*. Relative steepness we define as follows:

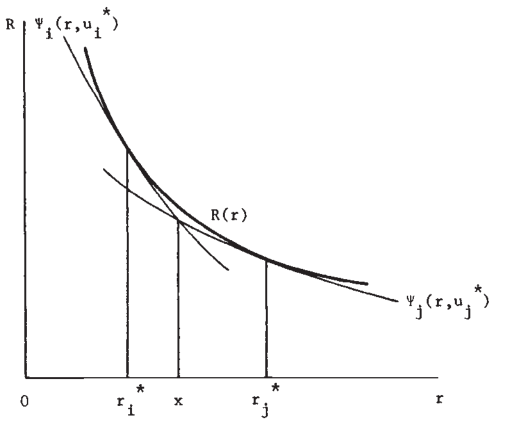

Figure 2.8. Ordering of equilibrium locations.

Definition 2.2. Suppose that bid rent functions Ψ_i and Ψ_j are continuous in r. Then we say that Ψ_i is *steeper* than Ψ_j if and only if the following condition is satisfied: Whenever $\Psi_i(x, u_i) = \Psi_j(x, u_j) > 0$ for some (x, u_i, u_j), then

$$\Psi_i(r, u_i) > \Psi_j(r, u_j) \qquad \text{for all } 0 \leq r < x$$

and

$$\Psi_i(r, u_i) < \Psi_j(r, u_j) \qquad \text{for all } r \text{ such that } r > x \text{ and } \Psi_i(r, u_i) > 0.$$

In other words, Ψ_i is steeper than Ψ_j if and only if the following condition is met: Whenever a pair of bid rent curves $\Psi_i(r, u_i)$ and $\Psi_j(r, u_j)$ intersects at a distance x, the former dominates the latter to the left of x and the latter dominates the former to the right of x. The important point is that this condition must be satisfied by every pair of bid rent curves. When bid rent curves are nonincreasing, Definition 2.2 can be restated in a simpler way as follows:

Definition 2.2′. Suppose that bid rent functions Ψ_i and Ψ_j are nonincreasing and differentiable in r. Then Ψ_i is steeper than Ψ_j if the following condition is met:[20] Whenever $\Psi_i(x, u_i) = \Psi_j(x, u_j) > 0$, then

$$-\frac{\partial \Psi_i(r, u_i)}{\partial r} > -\frac{\partial \Psi_j(r, u_j)}{\partial r} \qquad \text{at} \quad r = x.$$

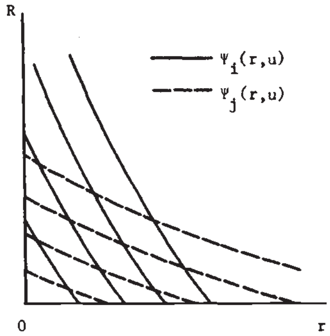

Figure 2.9. Relative steepness of bid rent functions.

That is, Ψ_i is steeper than Ψ_j if at the intersection of each pair of bid rent curves, apiece for households i and j, the former is always steeper than the latter (Figure 2.9).

It is obvious that if Ψ_i is steeper than Ψ_j, *no pair of bid rent curves intersects more than once (before reaching the r axis).* This means, in particular, that the equilibrium bid rent curve $\Psi_i(r, u_i^*)$ and the equilibrium bid rent curve $\Psi_j(r, u_j^*)$ intersect only once. Moreover, by definition, curve $\Psi_i(r, u_i^*)$ is steeper than curve $\Psi_j(r, u_j^*)$ at the intersection. Therefore, from Rule 2.2, we can state the following:

> ***Rule 2.3.*** If the bid rent function of household i is steeper than that of household j, the equilibrium location of household i is closer to the CBD than that of household j.

The applicability of this rule is limited in that we may not always be able to ascertain the relative steepness of bid rent functions among households. Nevertheless, we will see that it is very useful in comparative static analysis, where the effects of difference in model parameter values are examined. In fact, when a definite conclusion can be obtained from a comparative static analysis of household location, the relative steepness of bid rent functions (determined by parameter values) can almost always be ascertained. An important example is the effect of income level on household location.[21]

In the context of basic model (2.1), let us arbitrarily specify two income levels such that $Y_1 < Y_2$. It is assumed that both households possess the same utility function and face the same transport cost function. Denote by $\Psi_i(r, u)$ and $S_i(r, u)$ the bid rent and bid-max lot size functions of the household with income Y_i ($i = 1, 2$). Let us arbitrarily take a pair of bid rent curves $\Psi_1(r, u_1)$ and $\Psi_2(r, u_2)$, and suppose that they intersect at some distance x: $\Psi_1(x, u_1) = \Psi_2(x, u_2) \equiv \bar{R}$. Recall identity (2.19). Since $Y_1 - T(x) < Y_2 - T(x)$, from the normality of land,

$$S_1(x, u_1) = \hat{s}(\bar{R}, Y_1 - T(x)) < \hat{s}(\bar{R}, Y_2 - T(x)) = S_2(x, u_2).$$

Thus, from (2.27),

$$-\frac{\partial \Psi_1(x, u_1)}{\partial r} = \frac{T'(r)}{S_1(x, u_1)} > \frac{T'(r)}{S_2(x, u_2)} = -\frac{\partial \Psi_2(x, u_2)}{\partial r}.$$

Since we have arbitrarily chosen two bid rent curves, this result means that function Ψ_1 is steeper than Ψ_2. Thus, from Rule 2.3, we can conclude as follows:

> ***Proposition 2.1.*** Households with higher incomes locate farther from the CBD than households with lower incomes, other aspects being equal.

This result has often been used to explain the residential pattern observed in the United States.[22]

In closing this section, note that Proposition 2.1 was obtained through an examination of the way the steepness of a bid rent function changes with income. The same approach of examining the change in steepness of a bid rent function with respect to a parameter will often be used in the subsequent analysis. For this reason, it is helpful to introduce a mathematical operation that is useful for examining the change in relative steepness. Consider a general bid rent function $\Psi(r, u \mid \theta)$ with parameter θ. In order to examine how the relative steepness of function Ψ changes in θ, we arbitrarily choose a bid rent curve $\Psi(\cdot, u \mid \theta)$, and take a point $(r, \Psi(r, u \mid \theta))$ on that curve. Then *by keeping the value of* $\Psi(r, u \mid \theta)$ *constant,* we examine how the slope of that bid rent curve changes at r when parameter θ is changed (Figure 2.10). That is, we perform the following calculation:

$$-\frac{\partial \Psi_r(r, u \mid \theta)}{\partial \theta}\bigg|_{\Psi(r,u\mid\theta)=\text{const}}, \tag{2.39}$$

where $\Psi_r(r, u \mid \theta) \equiv \partial\Psi(r, u \mid \theta)/\partial r$. Operation (2.39) is often simply expressed as

$$-\frac{\partial \Psi_r}{\partial \theta}\bigg|_{d\Psi=0} \tag{2.40}$$

Then recalling Definition 2.2′, we can immediately conclude as follows:

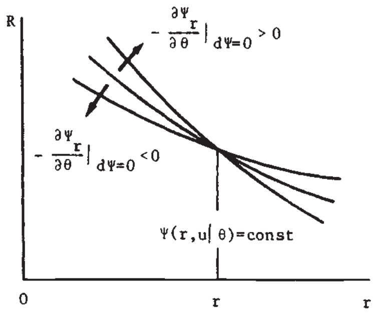

Figure 2.10. Change in the relative steepness of bid rent function $\Psi(r, u \mid \theta)$.

Rule 2.4. If $-(\partial\Psi_r/\partial\theta)\,|_{d\Psi=0}$ is positive (negative) at every point such that $\Psi(r, u \mid \theta) > 0$, then Ψ becomes steeper (less steep) as θ increases.

This rule is explained in Figure 2.10. As an illustration, let us derive Proposition 2.1 by applying this rule. In order to emphasize that Y is the parameter of interest, let us denote the bid rent and lot size functions obtained from (2.8) by $\Psi(r, u \mid Y)$ and $S(r, u \mid Y)$, respectively. Then from identity (2.19),

$$S(r, u \mid Y) = \hat{s}(\Psi(r, u \mid Y), Y - T(r)). \tag{2.41}$$

And from (2.27), $\Psi_r(r, u \mid Y) = -T'(r)/S(r, u \mid Y) = -T'(r)/\hat{s}(\Psi(r, u \mid Y), Y - T(r))$. Therefore, since $I = Y - T(r)$ at distance r,

$$-\frac{\partial\Psi_r}{\partial Y}\bigg|_{d\Psi=0} = \frac{\partial[T'(r)/\hat{s}(\Psi(r, u \mid Y), Y - T(r))]}{\partial Y}\bigg|_{\Psi(r,u|Y)=\text{const}}$$

$$= -\frac{T'(r)}{\hat{s}^2}\frac{\partial\hat{s}}{\partial I}\frac{\partial(Y - T(r))}{\partial Y}$$

$$= -\frac{T'(r)}{\hat{s}^2}\frac{\partial\hat{s}}{\partial I} < 0,$$

which is negative because $\partial\hat{s}/\partial I > 0$ from the normality of land. Since this result holds at any point such that $\Psi(r, u \mid Y) > 0$, from Rule 2.4 we can conclude that the bid rent function $\Psi(r, u \mid Y)$ becomes less steep as income increases. Therefore, Proposition 2.1 follows from Rule 2.3.

2.5 Extended models

Having mastered the basic model (2.1), it is appropriate to incorporate some of the important factors that we have previously neglected. In the first subsection, we introduce time cost in commuting and examine how the household's location is affected by wage income and nonwage income. In the second subsection, we examine the locational implications of family structure. In the third subsection, we study the so-called *Muth model,* in which the housing service is produced by the housing industry.

2.5.1 Time-extended model

Although we have not explicitly considered the time cost of commuting, in practice time cost is as important as pecuniary cost. In order to examine the effects of pecuniary cost and time cost on residential choice, we assume that the household will maximize its utility subject to a budget constraint and a time constraint. The utility function is specified as $U(z, s, t_l)$, where z and s are the same as before, and t_l represents the leisure time. Suppose the household chooses distance r from the CBD. Then the total available time $\bar{t}$ is spent on the leisure time t_l, the working time t_w, and the commuting time br, where b is a constant representing the commuting time per distance. Thus, the time constraint of the household is given as $t_l + t_w + br = \bar{t}$. The income of the household is the sum of nonwage income Y_N and wage income Wt_w, where W represents the wage rate. This total income is spent on composite good z, land rent $R(r)s$, and transport cost ar, where a is a constant representing the pecuniary commuting cost per distance. Hence, the budget constraint of the household is given as $z + R(r)s + ar = Y_N + Wt_w$. We assume that the household can freely choose its leisure time and working time. Then the residential choice of the household can be expressed as

$$\max_{r,z,s,t_l,t_w} U(z, s, t_l),$$

$$\text{subject to} \quad z + R(r)s + ar = Y_N + Wt_w \quad \text{and} \quad t_l + t_w + br = \bar{t}, \tag{2.42}$$

which is called the *time-extended model* of residential choice.

From the time constraint, $t_w = \bar{t} - t_l - br$. Substituting this into the budget constraint, the above model can be restated as[23]

$$\max_{r,z,s,t_l} U(z, s, t_l), \qquad \text{subject to} \quad z + R(r)s + Wt_l = I(r), \tag{2.43}$$

where $I(r) \equiv Y_N + I_w(r) - ar$, and $I_w(r) \equiv W(\bar{t} - br)$. This formulation suggests that the household makes the following transaction: It sells all available time (net of commuting, $\bar{t} - br$) to employers at the wage rate W; it then purchases back its leisure time t_l at the same unit price of time, W. Thus, wage rate W also serves as the *unit price of leisure time*. We may call $I_w(r)$ and $I(r)$ the *potential wage income* and the *potential net income* at distance r, respectively. At this point, it is also convenient to define

$$T(r) = ar + Wbr, \tag{2.44}$$

which represents the total commuting costs at distance r.

We assume that with obvious modifications Assumptions 2.1–2.3 hold for this time-extended model.[24] Then recalling Definition 2.1, the bid rent function for this model is stated as

$$\Psi(r, u) = \max_{z,s,t_l} \left\{ \frac{I(r) - z - Wt_l}{s} \,\middle|\, U(z, s, t_l) = u \right\}. \tag{2.45}$$

From this, the bid-max consumption bundle $(z(r, u), S(r, u), t_l(r, u))$ can be derived in a manner essentially identical to that in the case of the basic model. First, solving the utility constraint $U(z, s, t_l) = u$ for z, we obtain the equation of the indifference surface as $z = Z(s, t_l, u)$. Substituting this into (2.45), we obtain the unconstrained version of the bid rent function:

$$\Psi(r, u) = \max_{s,t_l} \frac{I(r) - Z(s, t_l, u) - Wt_l}{s}. \tag{2.46}$$

The first-order conditions for the optimal choice of (s, t_l) are

$$-\frac{\partial Z}{\partial s} = \Psi(r, u), \qquad -\frac{\partial Z}{\partial t_l} = W, \tag{2.47}$$

which express the familiar marginality conditions asserting that at the optimal choice of consumption bundle, the marginal rate of substitution between each pair of goods equals the corresponding price ratio (recall that the price of z equals 1).

Example 2.3. Suppose that the utility function in model (2.45) is given by the following log-linear function:

$$U(z, s, t_l) = \alpha \log z + \beta \log s + \gamma \log t_l, \tag{2.48}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\alpha + \beta + \gamma = 1$. Then $Z(s, u, t_l) = s^{-\beta/\alpha} t_l^{-\gamma/\alpha} e^{u/\alpha}$, and using (2.47) we obtain

$$\Psi(r, u) = \alpha^{\alpha/\beta}\beta(\gamma/W)^{\gamma/\beta}I(r)^{1/\beta}e^{-u/\beta}, \tag{2.49}$$

$$z(r, u) = \alpha I(r), \qquad t_l(r, u) = \gamma I(r)/W, \tag{2.50}$$

$$S(r, u) = \beta I(r)/\Psi(r, u) = \alpha^{-\alpha/\beta}(\gamma/W)^{-\gamma/\beta}I(r)^{-(\alpha+\gamma)/\beta}e^{u/\beta}. \tag{2.51}$$

As with the basic model, a number of useful identities can be obtained. In particular, the two most important are described below. Let us generally represent the unit price of leisure time by P_l.[25] We define the Marshallian demand $\hat{s}(R, P_l, I)$ for land from the solution of the following utility-maximization problem:

$$\max_{z,s,t_l} U(z, s, t_l), \qquad \text{subject to} \quad z + Rs + P_l t_l = I. \tag{2.52}$$

Then it holds identically that[26]

$$S(r, u) \equiv \hat{s}(\Psi(r, u), W, I(r)). \tag{2.53}$$

That is, the bid-max demand for land under utility u is just the Marshallian demand under land rent $\Psi(r, u)$ and leisure price W. Similarly, if we define the Hicksian demand $\tilde{s}(R, P_l, u)$ for land from the solution of the next expenditure minimization problem,

$$\min_{z,s,t_l} z + Rs + P_l t_l, \qquad \text{subject to} \quad U(z, s, t_l) = u, \tag{2.54}$$

then it holds identically that

$$S(r, u) \equiv \tilde{s}(\Psi(r, u), W, u). \tag{2.55}$$

That is, the bid-max demand for land at utility u is identical to the compensated demand at utility u under land rent $\Psi(r, u)$ and leisure price W.

With these identities just stated, we can use the same techniques as before to confirm that Properties 2.1 and 2.2 of bid rent curves also pertain to the time-extended model.[27] In addition, Rule 2.1′ (or Rule 2.1) can similarly be used to determine the equilibrium location. The marginal change in bid rent with respect to distance is, as before,

$$\Psi_r \equiv \frac{\partial \Psi(r, u)}{\partial r} = -\frac{T'(r)}{S(r, u)}, \tag{2.56}$$

where

$$T'(r) = a + Wb. \tag{2.57}$$

We are now ready to examine the effects of nonwage income and wage income on the household's location. First, the effect of nonwage income Y_N is essentially the same as that of income Y in the basic model. Substituting identity (2.53) into (2.56), $\Psi_r = -T'(r)/\hat{s}(\Psi(r, u), W, I(r))$. Applying the method developed at the end of Section 2.4, we have

$$-\left.\frac{\partial \Psi_r}{\partial Y_N}\right|_{d\Psi=0} = \left.\frac{\partial[T'(r)/\hat{s}(\Psi(r, u), W, I(r))]}{\partial Y_N}\right|_{\Psi(r,u)=\text{const}}$$

$$= -\frac{T'(r)}{\hat{s}^2}\frac{\partial \hat{s}}{\partial I}\frac{\partial I(r)}{\partial Y_N}$$

$$= -\frac{a + Wb}{\hat{s}^2}\frac{\partial \hat{s}}{\partial I} < 0,$$

which is negative, since income effect $\partial \hat{s}/\partial I$ is positive from the normality of land. From Rule 2.4, this means that bid rent function Ψ becomes less steep as Y_N increases. Therefore, from Rule 2.3, we can conclude as follows:

> ***Proposition 2.1'.*** Households with higher nonwage incomes locate farther from the CBD than households with lower nonwage incomes, other aspects being equal.

With Propositions 2.1 and 2.1', we can conclude that as long as the transport cost is independent of income level, the affluent live farther from the CBD than the less affluent.

The next logical question is, How does wage income influence the locational choice of the household? Since wage rate affects both the transport cost function and the demand for land, the overall effect is not simple. In order to examine the effect of the wage rate on the steepness of the bid rent function, from (2.56) we calculate

$$-\left.\frac{\partial \Psi_r}{\partial W}\right|_{d\Psi=0} = \left(\frac{1}{S}\frac{\partial T'}{\partial W} - \frac{T'}{S^2}\frac{\partial S}{\partial W}\right)_{d\Psi=0}$$

$$= \frac{T'}{SW}\left(\frac{\partial T'}{\partial W}\frac{W}{T'} - \frac{\partial S}{\partial W}\frac{W}{S}\right)_{d\Psi=0},$$

where $S = S(r, u)$ and $T' = T'(r)$. Therefore,

$$-\left.\frac{\partial \Psi_r}{\partial W}\right|_{d\Psi=0} \gtreqless 0 \quad \text{as} \quad \underbrace{\left.\frac{\partial T'}{\partial W}\frac{W}{T'}\right|_{d\Psi=0}}_{\substack{\text{wage elasticity}\\ \text{of marginal}\\ \text{transport cost}}} \gtreqless \underbrace{\left.\frac{\partial S}{\partial W}\frac{W}{S}\right|_{d\Psi=0}}_{\substack{\text{wage elasticity}\\ \text{of lot size}}}. \tag{2.58}$$

The issue, then, reduces to a question of elasticities.[28] Since $T'(r) = a + Wb$,

$$\left.\frac{\partial T'}{\partial W}\frac{W}{T'}\right|_{d\Psi=0} = \frac{\partial T'}{\partial W}\frac{W}{T'} = \left(1 + \frac{a}{bW}\right)^{-1} \tag{2.59}$$

A simple calculation yields[29]

$$\left.\frac{\partial S}{\partial W}\frac{W}{S}\right|_{d\Psi=0} = \eta\frac{I_w(r)}{I(r)} + \varepsilon, \tag{2.60}$$

where

$$\eta = \frac{\partial \hat{s}}{\partial I}\frac{I(r)}{\hat{s}}, \qquad \varepsilon = \frac{\partial \hat{s}}{\partial P_l}\frac{P_l}{\hat{s}}. \tag{2.61}$$

By definition, η represents the *potential-net-income elasticity of lot size* and ε the *cross-elasticity of lot size to the price of leisure time.*[30] Since land is a normal good, η is always positive. We assume that these elasticities are constant in the relevant range of analysis.[31] Substituting (2.59) and (2.60) into (2.58), we obtain the following:

Property 2.5. In the context of the time-extended model.

$$-\left.\frac{\partial \Psi_r}{\partial W}\right|_{d\Psi=0} \gtreqless 0 \quad \text{as} \quad f(r, W) \equiv \left(1 + \frac{a}{bW}\right)^{-1} - \left(\eta\frac{I_w(r)}{I(r)} + \varepsilon\right) \gtreqless 0, \tag{2.62}$$

where $I(r) = Y_N + I_w(r) - ar$ and $I_w(r) = W(\bar{t} - br)$, and a and bW are, respectively, the marginal pecuniary cost and the marginal time cost of commuting.

Since the elasticity difference $f(r, W)$ is generally a function of r and W, it is difficult to obtain general conclusions about the effects of wage changes on the steepness of the bid rent function. But let us consider the special case in which households are pure-wage earners (i.e., $Y_N = 0$) and pecuniary transport costs are negligible relative to time costs (i.e., $a = 0$).[32] Under these conditions,

$$f(r, W) = 1 - (\eta + \varepsilon).$$

Hence, with Property 2.5 and recalling Rules 2.3 and 2.4, we can state the following proposition:

Proposition 2.2. Given that households consist of pure-wage earners whose pecuniary transport costs are zero (i.e., $Y_N = 0$, $a = 0$), then[33]

(i) if $\eta + \varepsilon > 1$, the equilibrium location of the household moves out from the CBD with increasing wage rates;
(ii) if $\eta + \varepsilon < 1$, the equilibrium location of the household moves in toward the CBD with increasing wage rates;
(iii) if $\eta + \varepsilon = 1$, wage rates do not affect location.

In Japan, for example, pecuniary commuting costs are often paid by employers ($a = 0$). Hence, Proposition 2.2(ii) can be used to explain the general tendency in most large Japanese cities for wealthy households to live closer to the CBD than less affluent households (provided that condition $\eta + \varepsilon < 1$ holds, which is the most common case). In the United States, however, pecuniary commuting costs are not negligible,[34] and the Proposition 2.2 is inapplicable.

When pecuniary commuting costs are not negligible, we can reconsider relation (2.62) in light of the pure-wage earners ($Y_N = 0$), for whom the following holds:

$$\frac{I_w(r)}{I(r)} = \left(1 - \frac{ar}{W(\bar{t} - br)}\right)^{-1}.$$

This ratio is 1 at $r = 0$, and it increases as r increases. Hence, if $\eta + \varepsilon \geq 1$, then $f(r, W) < 0$ for all r, and we can conclude from (2.62) that

$$\text{if} \quad \eta + \varepsilon \geq 1, \qquad \text{then} \quad -\left.\frac{\partial \Psi_r}{\partial W}\right|_{d\Psi=0} < 0,$$

which implies that high-wage earners reside farther from the CBD than low-wage earners.[35]

If $\eta + \varepsilon < 1$, function $f(r, W)$ can assume both positive and negative values. For this case, observe that in a realistic range of parameters r, a, b, and W, the elasticity of ratio $I_w(r)/I(r)$ in W is very small relative to the comparable elasticity of a/bW; note also that $I_w(0)/I(0) = 1$, and the rate of increase of ratio $I_w(r)/I(r)$ in r is very small.[36] Hence, we can safely state that

$$f(r, W) \doteqdot f(W) \equiv \left(1 + \frac{a}{bW}\right)^{-1} - (\eta + \varepsilon). \qquad (2.63)$$

Assuming $0 < \eta + \varepsilon < 1$, the behavior of function $f(W)$ is depicted in Figure 2.11a. The wage elasticity of marginal transport cost $(1 + a/bW)^{-1}$ is increasing from 0 to 1, while the wage elasticity of lot size

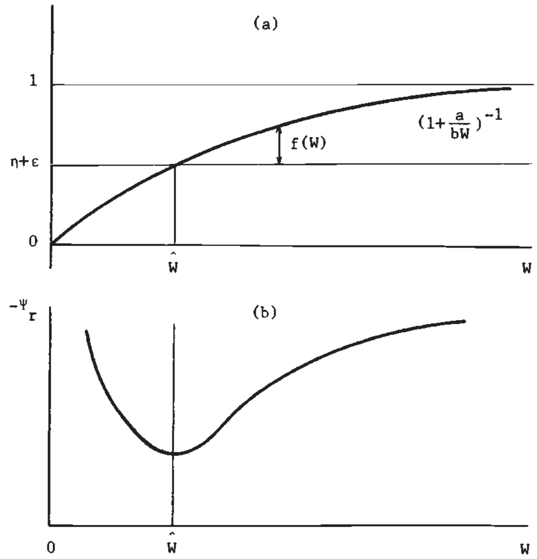

Figure 2.11. Effects of wage rate on the slope of the bid rent function ($\eta + \varepsilon < 1$).

stays constant at $\eta + \varepsilon$. Hence, the difference $f(W)$ is first negative for $W < \hat{W}$, then positive for $W > \hat{W}$, where

$$\hat{W} = \frac{a}{b}\frac{\eta + \varepsilon}{1 - (\eta + \varepsilon)}. \tag{2.64}$$

Thus, from Property 2.5, with changing wage rate, the slope of the bid rent curve ($-\Psi_r$) at each rent-location point varies as depicted in Figure 2.11b. It follows from Rules 2.3 and 2.4 that the equilibrium location of the household moves out from the CBD as the wage rate increases to $\hat{W}$, after which it moves in toward the CBD while the wage rate continues to increase.[37] This makes intuitive sense as follows. For low-income households, pecuniary transport costs are crucial, as are wages lost in time spent for commuting. For these reasons, low-income households tend to locate near the CBD. As incomes rise, such costs are less important, so households locate farther from the CBD. At some high wage rate, however, the opportunity cost of time spent for commuting becomes very significant. Households with such high wage rates tend to shift their locations back toward the city center. In short, we can conclude the following:

Proposition 2.3. Given that households consist of pure-wage earners whose pecuniary transport costs are positive (i.e., $Y_N = 0$, $a > 0$), then

(i) if $\eta + \varepsilon \geq 1$, the equilibrium location of the household moves farther from the CBD with increasing wage rates;
(ii) if $0 < \eta + \varepsilon < 1$, increases in the wage rate first move the equilibrium location away from the CBD; but beyond the wage rate $\hat{W}$, which is given by (2.64), additional increases retract the household location again.

When the substance of Proposition 2.3(ii) holds, both those who earn very low wages and those who earn very high wages tend to reside near the city center; middle-wage earners gravitate toward the suburbs. This is consistent with what has been observed in large cities in the United States.[38] The behavior of the curve in Figure 2.11b is also consistent with the estimate of slopes of bid rent curves in San Francisco by Wheaton (1977).

In closing this subsection, we note that Proposition 2.3 yields an important policy implication. Regardless of the wage elasticity of lot size ($\eta + \varepsilon$), low-income households will always prefer central locations. Thus, the demolition of low-quality housing in city centers does not induce a more even distribution of income classes throughout the entire city. It merely displaces a certain group, which will continue to seek a central location.[39]

2.5.2 *Family-structure model*

We now extend the model of Section 2.5.1 in order to encompass the effects of family structure on the locational decision. Following Beckmann (1973), we assume that the family structure of a household is characterized by two parameters: d, the number of dependent members, and n, the number of working members in the household. The utility function of the household is now generalized as $U(z, s, t_l; d, n)$, d and n being parameters. Thus, model (2.42) becomes

$$\max_{r,z,s,t_l,t_w} U(z, s, t_l; d, n),$$

$$\text{subject to} \quad z + R(r)s + nar = Y_N + nWt_W \quad \text{and} \quad t_l + t_w + br = \bar{t}, \tag{2.65}$$

which is called the *family-structure model* of residential choice. The second constraint represents the time constraint for *each* working member. Here all working members of the household are assumed to have the same leisure time t_l, working time t_w, and commuting time br. The first constraint represents the *family* budget. Each working member of the household is also assumed to have the same pecuniary transport cost ar, as well as the same wage rate W. Composite good z and land s are consumed in aggregate by all household members.

As in Section 2.5.1 we can rewrite the family-structure model as follows:

$$\max_{r,z,s,t_1} U(z, s, t_1; d, n),$$

$$\text{subject to} \quad z + R(r)s + nWt_1 = I(r, n), \tag{2.66}$$

where $I(r, n) = Y_N + nW(\bar{t} - br) - nar$. Thus, the bid rent function is now given by

$$\Psi(r, u) = \max_{s,t_1} \frac{I(r, n) - Z(s, t_1, u; d, n) - nWt_1}{s}, \tag{2.67}$$

where $Z(s, t_1, u; d, n)$ is the solution of $U(z, s, t_1; d, n) = u$ for z.

As an example, let us consider the case of the next log-linear utility function,

$$U(z, s, t_1; d, n) = h\alpha \log (z/h^{\lambda}) + h\beta \log (s/h^{\mu}) + n\gamma \log t_1 + d\delta \log \bar{t}, \tag{2.68}$$

where each of α, β, γ, δ, λ, and μ is a positive constant, and $h = d + n$ represents the family size.[40] For example, $\lambda = \mu = 1$ means that all family members equally share z and s. In practice, μ will be less than unity (reflecting the public-good nature of z and s for the household members). By direct calculation, the bid rent function and the bid-max consumption can be obtained as follows:

$$\Psi(r, u) = A\left(\frac{h\alpha}{B}\right)^{\alpha/\beta}\left(\frac{h\beta}{B}\right)\left(\frac{n\gamma}{B}\frac{1}{nW}\right)^{n\gamma/h\beta} I(r, n)^{B/h\beta} e^{-u/h\beta},$$

$$z(r, u) = \frac{h\alpha}{B} I(r, n), \qquad S(r, u) = \frac{h\beta}{B}\frac{I(r, n)}{\Psi(r, u)},$$

$$t_1(r, u) = \frac{n\gamma}{B}\frac{I(r, n)}{nW},$$

where $A = \{h^{h\alpha\lambda + h\beta\mu}(\bar{t})^{-d\delta}\}^{-1/(h\beta)}$ and $B = h\alpha + h\beta + n\gamma$. A simple calculation yields

$$-\frac{\partial \Psi}{\partial r} = \frac{\alpha + \beta + (n/h)\gamma}{\beta} \frac{a + bW}{(Y_N/n) + W(\bar{t} - br) - ar} \Psi(r, u). \tag{2.69}$$

Then since $h = d + n$,

$$-\frac{\partial \Psi_r}{\partial d}\bigg|_{d\Psi=0} = -\frac{n\gamma}{\beta h^2} \frac{a + bW}{(Y_N/n) + W(\bar{t} - br) - ar} \Psi(r, u) < 0,$$

which means that *the bid rent function becomes less steep with an increasing number of dependents*. Increasing d augments the weight on lot size in the utility function relative to the weight on leisure time of working members. This in turn increases the demand for lot size, and therefore the bid rent function becomes less steep. Next, we can see from (2.69) that *in the case of pure-wage earners* ($Y_N = 0$), *the bid rent function becomes steeper with an increasing* n/h, *the commuter–family size ratio*. Similarly, we can see from (2.69) that *in the case of pure-wage earners with no dependents* ($d = 0$, *and hence* $n/h = 1$), *the steepness of the bid rent function is independent of family size (= the number of commuters)*. Therefore, recalling Rule 2.3, we can conclude as follows:

Proposition 2.4. In the context of the family-structure model with a log-linear utility function, we have that

(i) the more dependents a household has, the farther is its equilibrium location from the CBD;
(ii) given that households consist of pure-wage earners, locations can be ranked by the households' commuter–family size ratio n/h; the smaller the ratio, the farther is the location from the CBD;
(iii) given that households consist of pure-wage earners with no dependents, locations are independent of family size (i.e., the number of commuters).

These conclusions, first obtained by Beckmann (1973), are consistent with many casual observations from U.S. cities. Although these conclusions result from a log-linear utility function, it is not difficult to obtain similar conclusions from the original model of (2.65).

2.5.3 *Muth model of housing industry*

In the basic model of (2.1), it is implicitly assumed that each household manages the construction of its house by itself. There is, however, another class of models, originated by Muth (1969), in which households are assumed to consume an aggregate commodity called the *housing service*. That is, each household behaves as

$$\max_{r,z,q} U(z, q), \qquad \text{subject to} \quad z + R_H(r)q = Y - T(r), \tag{2.70}$$

where $R_H(r)$ is the unit price of housing service q at location r, and z represents the amount of composite consumer good excluding housing service. In turn, the housing industry produces the housing service with production function $F(L, K)$ from land L and capital (or nonland input) K. That is, each profit-maximizing firm of the housing industry behaves as

$$\max_{L,K} R_H(r)F(L, K) - R(r)L - K, \qquad \text{at each} \quad r, \tag{2.71}$$

where $R(r)$ is the land rent at r, and the price of capital, which is assumed to be a fixed constant independent of location, is normalized to unity.

When combined, (2.70) and (2.71) can be called the Muth model of the housing industry. There are two different ways to treat this model. One is to reformulate it as a version of the basic model. Let q be the amount of housing service consumption by a household, and define

$$s \equiv \frac{q}{F(L, K)} L, \qquad k \equiv \frac{q}{F(L, K)} K. \tag{2.72}$$

Then, s and k represent, respectively, the land input and capital input per household. Let us assume, as in Muth (1969), that the housing production function F has constant returns to scale. Then a simple calculation yields[41]

$$q = F(s, k), \tag{2.73}$$

which represents the housing production function in terms of inputs and output per household. Again, since F has constant returns to scale, in equilibrium the housing industry gets zero profit at each location: $R_H(r)F(L, K) - R(r)L - K = 0$. Hence,

$$\begin{aligned} R_H(r) &= R(r)L/F(L, K) + K/F(L, K) \\ &= R(r)s/q + k/q. \end{aligned} \tag{2.74}$$

Substituting (2.73) and (2.74) into (2.70), the Muth model is equivalent to the following *reduced-form model,* in which each household chooses land and capital inputs by itself:

$$\max_{r,z,s,k} U(z, F(s, k)), \qquad \text{subject to} \quad z + k + R(r)s = Y - T(r). \tag{2.75}$$

Except for the addition of a new choice variable k, this is essentially the same as the basic model.[42]

Another way is to keep the context of the Muth model, which is more appropriate for the study of apartment-type houses. Let us define the *bid housing rent function* $\Psi_H(r, u)$ as

$$\Psi_H(r, u) = \max_q \frac{Y - T(r) - Z(q, u)}{q}, \tag{2.76}$$

where $Z(q, u)$ is the solution of $u = U(z, q)$ for z. Note that except for notational differences, this is the same as (2.8). Therefore, if we replace $R(r)$ and $\Psi(r, u)$, respectively, with $R_H(r)$ and $\Psi_H(r, u)$, then all the results of the previous sections hold true for the Muth model. Specifically, let us assume that Assumptions 2.1–2.3 hold when s is replaced with q. Then, Propositions 2.1–2.4 can also be derived from the Muth model.[43] In this sense, they represent very robust conclusions. We will continue discussion of the Muth model in Section 3.7.

2.6 Conclusion

In this chapter, we have examined the residential choice of the household as determined by the trade-off between space for living and accessibility to work. We began with the basic model, in which only pecuniary transport costs were explicitly considered. Then we introduced the time cost of commuting, family structure, and housing consumption.

Our models produced results which suggest that a particular land use pattern will prevail in the monocentric city. Suppose the pecuniary transport costs are not negligible and the wage elasticity of lot size is less than unity. Then according to Propositions 2.1, 2.3, and 2.4, the following land use pattern will prevail. Wage-poor and wage-rich households with few dependents (such as singles and working couples with few children) will tend to reside close to the city center. Beyond them and out toward the suburbs, middle-income households with large families and few commuters will be found. Farther away, asset-rich households with larger families and few commuters will locate. This pattern is consistent with what has been observed of large cities in the United States.

Recall that all the propositions of this chapter have been obtained by the same, simple method of analysis. That is, we examined how the steepness of the bid rent function changed with the change in parameter values. If the bid rent function becomes steeper with an increasing parameter value, the households with greater parameter values will locate closer to the city center than will those with smaller parameter values and vice versa. Note that our analysis made no assumptions about the shape of the market rent curve or about the behavior of landowners except to assume that households are price takers who see the market land rent curve as an exogenous factor. Therefore, these conclusions about the land use pattern hold irrespective of the shape of the market land rent curve and the behavior of landowners.

However, if we want more detailed information about the equilibrium land use pattern, such as population density and the shape of the market rent curve, we must, of course, specify the behavior of landowners too. We will do this in the next chapter.

Bibliographical notes

The theory of household location presented in this chapter is derived in large part from the pioneering work of Alonso, Beckmann, and Muth. The basic model of Section 2.2 is a simplification of Alonso's model (1964, Ch. 2). In the original Alonso model, utility function includes another variable, the distance to the CBD, which is supposed to represent the disutility of commuting. However, with the Alonso model it is hard to obtain any general result on the location of the household. Therefore, in order to obtain clear-cut results, most subsequent works adopted the simpler framework of the basic model.

The bid rent function approach described in this chapter was first established by Alonso (1964). This is, of course, a generalization of the agricultural bid rent theory of von Thünen (1826). This approach is essentially the same as the indirect utility function approach introduced into an urban land use model by Solow (1973). Many urban economists, notably Schweizer, Varaiya, and Hartwick (1976) and Kanemoto (1980), further developed this bid rent/indirect utility function approach. The concept of relative steepness of bid rent functions was introduced by Fujita (1985).

The time-extended model of Section 2.5.1 is an extension of similar models by Beckmann (1974), Henderson (1977), and Hochman and Ofek (1977), which consider only the time cost of commuting, neglecting the pecuniary cost. Our discussion of this extended model is based on Fujita (1986a). A similar model was independently studied by DeSalvo (1985). Proposition 2.2 is essentially the same as Corollary 3 of Hochman and Ofek (1977). We can also consider the time-extended model as a simplified version of Yamada (1972). In Yamada's work, other factors such as the disutility of working time and commuting time and environmental external effects are also considered. Note that here the household can freely choose the length of working time. For the case in which maximum working length is considered, see Moses (1962) and Yamada (1972).

The family-structure model of Section 2.5.2 is an extension of Beckmann (1973). In Beckmann's model, the pecuniary transport cost is assumed to be zero and the working time is fixed. The housing industry model of Section 2.5.3 was, of course, introduced by Muth (1969). In Muth's study, transport cost is implicitly assumed to be a function of income level. In our model, wage income and nonwage income are treated separately, and hence pecuniary transport costs are assumed to be independent of income.

In this chapter, in order to explain the general pattern of household location observed in the United States, we have focused mainly on the time-extended model. LeRoy and Sonstelie (1983) present an alternative model that introduces multiple transport modes.

Notes

1. For an introduction to the consumer theory relevant to the following discussion, the reader is referred, e.g., to J. M. Henderson and R. E. Quandt (1980) and Varian (1984). See also Appendix A.3 for a summary of important results from consumer theory.
2. This utility function is simple, yet general enough to serve our present purpose of focusing on lot size and household density changes in the city. The function that appears in the text was derived as follows: First assume that the original utility function of the household is given by $U(z_1, \ldots, z_n, s)$, where each z_i ($i = 1, 2, \ldots, n$) represents the amount of consumer good i (other than land), and s the lot size. Some of the z_i's represent nonland inputs for housing. Assuming that the price of each consumer good i does not vary within the city; we represent it by p_i, $i = 1, 2, \ldots, n$ (this assumption is appropriate because compared with land rent, the prices of other goods are relatively constant within a city). Under each fixed combination of (z, s), define

$$U(z, s) = \max\left\{ U(z_1, \ldots, z_n, s) \;\middle|\; \sum_1^n p_i z_i = z \right\},$$

   where z represents the total expenditure for all consumer goods other than land. This derived utility function is the one in the text (refer to the Aggregation theorem of Hicks 1946, pp. 312–13). For alternative specifications of the utility function, see Section 2.5.
3. For some of the mathematical terminology used in the following discussion (e.g., strictly convex and smooth curves), see Appendix A.1.
4. Strictly speaking, the fact that the utility function is increasing in z and s implies that condition (2.3) holds *almost everywhere*. That is, it cannot rule out the possibility that $\partial U/\partial z$ or $\partial U/\partial s$ becomes zero on a set of points with measure zero. However, this minor difference does not affect our results in any essential way, and hence we neglect it in the following discussion. The same note applies to conditions (2.5) and (2.6).
5. Formally, we have a constraint $s > 0$. However, since any indifference curve does not cut axes, whenever it exists, the optimal s for the maximization problem of (2.8) is positive (Figure 2.2). Hence, we can neglect the positivity constraint on s. Note also that given s, it may not be possible to solve

the equation $u = U(z, s)$ for z. In this case, we define $Z(s, u) = \infty$. Then in the maximization problem of (2.8), such s will never be chosen as the optimal lot size.

6. Since each indifference curve is strictly convex, if it exists, the bid-max lot size is unique (Figure 2.2). We introduce the following convention: When there is no solution to the maximization problem of (2.8), we define $\Psi(r, u) = 0$ and $S(r, u) = \infty$. Note also that when we solve the maximization problem of (2.7) or (2.8), we also obtain the *bid-max composite good consumption* $z(r, u) \equiv Z(S(r, u), u)$. However, since we will never use function $z(r, u)$ in the subsequent analysis, we omit its discussion.
7. More precisely, if we denote the angle ACO by θ, then $\Psi(r, u) = \tan\theta$. But for simplicity, we use this graphical expression throughout the book.
8. In detail, we have

$$\frac{\partial}{\partial s}\left(\frac{Y - T(r) - Z(s, u)}{s}\right) = -\frac{1}{s}\frac{\partial Z(s, u)}{\partial s} - \frac{Y - T(r) - Z(s, u)}{s^2} = 0,$$

which leads to (2.10).

9. Using (2.10),

$$\frac{\partial^2}{\partial s^2}\left(\frac{Y - T(r) - Z(s, u)}{s}\right) = -\frac{1}{s}\frac{\partial^2 Z(s, u)}{\partial s^2} < 0$$

from (2.5). This implies that function $(Y - T(r) - Z(s, u))/s$ is strictly concave in s, and hence the first-order condition (2.10) gives the necessary and sufficient condition for optimal s.

10. For the actual calculation, see Appendix C.1.
11. More generally, the concept of bid rent is useful in any market where buyers choose one good (or at most a few) from a family of highly substitutable goods. Examples are automobiles and housing.
12. Note that under each value of E, the equation $E = z + \Psi(r, u)s$ represents an expenditure line (i.e., budget line), which is parallel to line AC in Figure 2.2. This expenditure line shifts upward with increasing E. Hence, $Y - T(r) = z + \Psi(r, u)s$ gives the lowest expenditure line under which utility level u is attainable.
13. See Appendix A.3 for a summary of important characteristics of demand functions and related functions.
14. For the envelope theorem, see Appendix A.2.
15. In Figure 2.5, bid rent curves are depicted as intercepting the r axis at different points. This is not always true, however. For example, in the case of a log-linear utility function (Example 2.1), we see that all bid rent curves intercept the r axis at distance $\bar{r}$ defined as $Y - T(\bar{r}) = 0$. A shared interception point occurs *if land is completely substitutable for the composite good;* that is, if the utility function from which the bid rent curves are derived is imbued with the feature that for every u, indifference curve $Z(s, u)$ approaches the s axis as $s \to \infty$. However, this minor difference in the shape

of bid rent curves does not cause any important difference in the subsequent analysis.

16. The terms *equilibrium location* and *optimal location* are often used interchangeably. Note that *optimal* simply means the best for the household, and it does not imply any social value judgment.
17. This statement is intuitively appealing, but not very precise. Since the optimal location of the household may not be unique, it is more accurate to call the equilibrium location an optimal location. Also, tangency must be interpreted broadly so as to include the possibility of a corner solution. Finally, if more than one bid rent curve is tangent to $R(r)$, we must choose the lowest among the curves.
18. We may assume that the residential choice behavior of household i is described as

$$\max_{r,z,s} U_i(z, s), \qquad \text{subject to} \quad z + R(r)s = Y_i - T_i(r)$$

and that of household j as

$$\max_{r,z,s} U_j(z, s), \qquad \text{subject to} \quad z + R(r)s = Y_j - T_j(r),$$

where U_i, Y_i, T_i are, respectively, utility function, income, and transport cost function of household i, and U_j, Y_j, T_j are those of j. Then we can derive the bid rent function $\Psi_i(r, u)$ of household i and $\Psi_j(r, u)$ of household j as explained in Section 2.3. However, the following rules (including Rule 2.1) are valid regardless of the specifications of residential choice behaviors from which bid rent functions have been derived. Hence, we simply assume that these bid rent functions have been derived from some residential choice models.

19. An exception may occur when $R(r)$ is kinked at x. In this case both households may possibly reside at x. But for any shape of market rent curve, it never happens that the household i with steeper equilibrium bid rent curve resides to the right of household j.
20. In order for one to say "if and only if," the following condition must be changed as follows: Whenever $\Psi_i(x, u_i) = \Psi_j(x, u_j)$, then $-\partial\Psi_i(r, u_i)/\partial r > -\partial\Psi_j(r, u_j)/\partial r$ at $r = x$, or $\partial\Psi_i(r, u_i)/\partial r = \partial\Psi_j(r, u_i)/\partial r$ and $\partial^2\Psi_i(r, u_i)/\partial r^2 < \partial^2\Psi_j(r, u_j)/\partial r^2$ at $r = x$.
21. For other examples, see Sections 2.5.1 and 2.5.2.
22. This result depends critically on the assumptions that all households have the same utility function and that transport costs are independent of income. A completely reversed spatial pattern can be observed in many European, Latin American, and Asian cities. In the United States as well, luxury apartments and townhouses are often found near the urban center. See Alonso (1964, Ch. 6), Muth (1969), and Wheaton (1977) for empirical studies of household location. These observations suggest that factors other than income, such as the time cost of commuting, family structure, externalities,

and dynamic factors, also affect residential choices and spatial patterns. These factors will be introduced one by one in the rest of the book.

23. In the following, we assume that t_w is always positive at the optimal choice (i.e., the household is not living retired); and hence we neglect the non-negativity constraint on t_w.
24. Specifically, Assumption 2.1 is changed as follows: The utility function is continuous and increasing at all $z > 0$, $s > 0$, and $t_l > 0$; and all indifference surfaces are strictly convex and smooth, and do not cut axes. Assumption 2.2 is simply changed as $a > 0$, $b > 0$. Assumption 2.3 remains as it is. Finally, it is assumed that the utility function is twice continuously differentiable, having no singular point.
25. The price of leisure time means the opportunity cost of leisure time. In our model, of course, it happens to be $P_l = W$. Here, we treat the price of leisure time as a parameter represented by P_l.
26. This can easily be seen because the two conditions of (2.47) also represent the optimality conditions for the problem (2.52) with $R = \Psi(r, u)$, $P_l = W$, and $I = I(r)$. Similar arguments apply to identity (2.55).
27. Since the transport cost function $T(r) = ar + Wbr$ is linear, Property 2.2 also holds trivially.
28. $(\partial T'/\partial W)(W/T') = (\partial T'/T')/(\partial W/W)$, which represents the percent change in marginal transport cost with respect to the percent increase in wage rate. Similarly, $(\partial S/\partial W)(W/S) = (\partial S/S)/(\partial W/W)$, which represents the percent change in lot size with respect to the percent increase in wage rate.
29. From (2.53), $S(r, u) = \hat{s}(\Psi(r, u), W, I(r))$, and $P_l = W$ by definition. Hence,

$$\frac{\partial S}{\partial W}\frac{W}{S}\bigg|_{d\Psi=0} = \left(\frac{\partial \hat{s}}{\partial I}\frac{\partial I(r)}{\partial W} + \frac{\partial \hat{s}}{\partial P_l}\frac{\partial P_l}{\partial W}\right)\frac{W}{\hat{s}} = \left(\frac{\partial \hat{s}}{\partial I}(\bar{t} - br) + \frac{\partial \hat{s}}{\partial P_l}\right)\frac{W}{\hat{s}}$$

$$= \frac{\partial \hat{s}}{\partial I}\frac{I_w(r)}{\hat{s}} + \frac{\partial \hat{s}}{\partial P_l}\frac{W}{\hat{s}} = \left(\frac{\partial \hat{s}}{\partial I}\frac{I(r)}{\hat{s}}\right)\frac{I_w(r)}{I(r)} + \frac{\partial \hat{s}}{\partial P_l}\frac{P_l}{\hat{s}}.$$

30. What we can observe in the market is not η, but the *realized-net-income elasticity of lot size* defined as

$$\eta_R = \frac{\partial \hat{s}}{\partial I_R(r)}\frac{I_R(r)}{\hat{s}},$$

where $I_R(r)$ is the realized net income at location r given as $I_R(r) = I(r) - Wt_l(r, u)$. The relation between η and η_R can be obtained as follows: Let $\hat{t}_l(R, P_l, I)$ be the ordinary demand for leisure time, which is obtained from the solution of the utility-maximization problem of (2.52). Then it immediately follows that $t_l(r, u) = \hat{t}_l(\Psi(r, u), W, I(r))$, and hence $I_R(r) = I(r) - W\hat{t}_l(\Psi(r, u), W, I(r))$. So

$$\frac{\partial \hat{s}}{\partial I} = \frac{\partial \hat{s}}{\partial I_R}\frac{\partial I_R}{\partial I} = \frac{\partial \hat{s}}{\partial I_R}\left(1 - W\frac{\partial \hat{t}_l}{\partial I}\right).$$

Hence,

$$\eta = \frac{\partial \hat{s}}{\partial I}\frac{I(r)}{\hat{s}} = \frac{\partial \hat{s}}{\partial I_R}\left(1 - W\frac{\partial \hat{t}_l}{\hat{s}}\right)\frac{I(r)}{\hat{s}}$$

$$= \frac{\partial \hat{s}}{\partial I_R}\frac{I_R(r)}{\hat{s}}\left(1 - W\frac{\partial \hat{t}_l}{\partial I}\right)\frac{I(r)}{I_R(r)} = \eta_R\left(1 - W\frac{\partial \hat{t}_l}{\partial I}\right)\left(1 + W\frac{\hat{t}_l}{I_R(r)}\right).$$

That is,

$$\eta = \eta_R\left(1 - W\frac{\partial \hat{t}_l}{\partial I}\right)\left(1 + W\frac{\hat{t}_l}{I_R(r)}\right).$$

31. When this assumption does not hold, we must read Propositions 2.2 and 2.3 with care. For this point, see note 33.
32. This simplifying assumption is often adopted in urban economics (e.g., Beckmann 1974; Henderson 1977; Hochman and Ofek 1977).
33. When η and ε are not constant, we must read, e.g., (i) as follows: "(i) if $\eta + \varepsilon > 1$ *in the relevant range of the analysis*, the equilibrium location of. . . ." The same note applies to Proposition 2.3.
34. E.g., Altmann and DeSalvo (1981) estimate that for the period 1960–75, the value of the ratio a/bW for an urban household with average income was equal to 0.9. Mills (1972a, p. 85) uses a value of $a/bW = 0.6$. It is reasonable to assume that this ratio is even greater now since the oil price increases in 1973.
35. An example is the case of the log-linear utility function (Example 2.3), for which $\eta = 1$ and $\varepsilon = 0$. In fact, from (2.49), $-\Psi_r = I'(r)\Psi(r, u)/\beta I(r) = T'(r)\Psi(r, u)/\beta I(r) = (a + Wb)\Psi(r, u)/\beta I(r)$. So assuming $Y_N = 0$,

$$-\left.\frac{\partial \Psi_r}{\partial W}\right|_{d\Psi=0} = -\frac{a\bar{t}\Psi(r, u)}{\beta I(r)^2} < 0.$$

36. The elasticity of a/bW in W is -1, while the elasticity of $I_w(r)/I(r)$ in W is $-x(W)/(1 - x(W))$, where $x(W) = ar/W(t - br)$. If we use parameter values from Altmann and DeSalvo (1981), we have $b = 1$ (round trip)/35 miles/hour = 1/17.5 (miles/hours), and $a = 1$ (round trip) $\times$ 4.61 (cents/mile $\cdot$ car) = 0.0922 (dollars/mile $\cdot$ car). Let us set $\bar{t}$ equal to 24 hours and r equal to 50 miles, which is more than the radius of the largest city today. Then the elasticity of $I_w(r)/I(r)$ in W is $0.22/(W - 0.22)$, which is close to zero under any reasonable value of wage rate (dollars/hour) in the United States. Similarly, $I_w(0)/I(0) - I_w(50)/I(50) \doteqdot 1 - 1/(1 - 4.6/21W)$, which is also near zero for U.S. wage rates.
37. If we use parameter values from Altmann and DeSalvo (1981) once again, then $\hat{W} = (a/b)(\eta + \varepsilon)/(1 - (\eta + \varepsilon)) = \$1.61(\eta + \varepsilon)/(1 - (\eta + \varepsilon))$, and $\eta_R = 0.875$. The last equation in note 30 suggests that η will be close to η_R. Hence, if we assume that $\eta = \eta_R = 0.875$ and $\varepsilon = 0$, we have $\hat{W} = \$11.27$/hour. So the annual wage income = \$11.27 $\times$ 40 hours/week $\times$

50 weeks = \$22,540/year · worker. If we adjust this number by average number of workers per household and by nonwage incomes, we obtain a considerably higher value than mean urban household income, which was \$12,577 in 1970.

38. Recent studies in the United States indicate that the wage elasticity of lot size, $\eta + \varepsilon$, may be considerably less than unity. Sample values from the literature for the realized-gross-income elasticity of housing are 0.75 cited by Muth (1971), 0.5 cited by Carliner (1973), and 0.75 cited by Polinsky (1977). Wheaton (1977) estimates the realized-gross-income elasticity of land to be 0.25. Since the value of ε will be close to zero, these numbers suggest that $\eta + \varepsilon$ may be considerably less than unity.
39. This is the point emphasized by Muth (1969).
40. When (2.66) and (2.68) are combined, the model represents an extension of Beckmann (1973) in which pecuniary commuting cost has been added.
41. From the first equation of (2.72), $q = F(L, K)s/L = F(s, Ks/L)$ (from the assumption of constant returns to scale) $= F(s, (kF(L, K)/q)(s/L))$ [from the second equation of (2.72)] $= F(s, (F(L, K)/Lq)(ks)) = F(s, (1/s)(ks)) = F(s, k)$.
42. Mathematically, this reduced model can be considered to be a special case of the basic model. That is, let us put $c = z + k$, and define $U(c, s) = \max_{z,k}\{U(z, F(s, k)) \mid z + k = c\}$. Then (2.75) is equivalent to the following: $\max_{r,c,s} U(c, s)$, subject to $c + R(r)s = Y - T(r)$. If we further replace c with z, we have the basic model.
43. In order to derive Propositions 2.2–2.4 from the Muth model, we must, of course, replace (2.70) with (2.42) or (2.65), in which s and $R(r)$ are replaced by q and $R_H(r)$, respectively.