ENPC - Operations Research and Transport - 2018

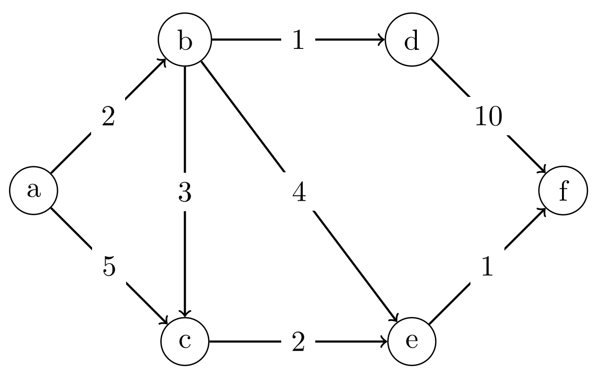
You have 2.5 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercice 1 (2pts). *Consider a game where rewards (to be maximized) are given by the following table where actions of player* 1 *correspond to the lines, actions of player* 2 *to the columns, rewards being given in the order of player. For example, if player* 1 *play* a, *and player* 2 *play* c, *then player* 1 *gains* 2 *and player* 2 *gains* 3.

	a	b	c
a	(7,1)	(0,0)	(2,3)
b	(-1,2)	(2,3)	(3,2)
c	(-1,4)	(1,3)	(1,7)

1. *Find the Nash equilibrium(s)*
2. *Find the social optimum(s)*
3. *Find the Pareto optimum(s)*

Exercice 2 (5pts). *Consider the following weighted graph.*



1. *Use Dijkstra's algorithm to find the cost of the shortest path between node a and node f. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm. Note the order in which the nodes are treated.*
2. *We have the following heuristic h giving an estimate of the distance between a given node and f.*

a	b	c	d	e	f
20	4	7	6	0	0

Apply the A^* algorithm using this heuristic. Note the order of nodes treated. Comment.

Exercice 3 (8pts). *Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k\in[\![1,K]\!]}$. We denote by (G, ℓ, r) the congestion game where*

- r^k *is the intensity of the flow of users entering in* o^k *and exiting in* d^k*;*
- $\mathcal{P}_k$ *is the set of all simple (i.e. without cycle) paths from* o^k *to* d^k*, and by* $\mathcal{P} = \bigcup_{k=1}^K \mathcal{P}_k$ *;*
- f_p *the number of users taking path* $p \in \mathcal{P}$ *per hour (intensity);*
- $f = \{f_p\}_{p\in\mathcal{P}}$ *the vector of path intensity;*
- $x_e = \sum_{p\ni e} f_p$ *the flux of user taking the arc* $e \in E$*;*
- $x = \{x_e\}_{e\in E}$ *the vector of arc intensity;*
- $x(f)$ *is the vector of edge-intensity induced by the path intensity* f*;*
- $\ell_e : \mathbb{R} \to \mathbb{R}^+$ *the cost incurred by a given user to take edge* e*, if the edge-intensity is* x_e*;*
- $L_e(x_e) := \int_0^{x_e} \ell_e(u)du$.

We want to compare the cost of the user equilibrium of (G, ℓ, r)*, denoted* $f^{UE,r}$*, with the cost of the social optimum* $f^{SO,2r}$ *of* $(G, \ell, 2r)$*, that is the same game with twice the inflows. Accordingly we denote* $x^{UE,r} = x(f^{UE,r})$*, and* $x^{SO,2r} = x(f^{SO,2r})$*. Finally, edge-loss* ℓ_e *are assumed to be non-negative and non-decreasing.*

We construct new loss functions $\bar{\ell}_e(x)$ *given by*

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(x^{UE,r}) & \text{if } x \leq x^{UE,r} \\ \ell_e(x) & \text{else} \end{cases}$$

Accordingly we denote $\bar{\ell}_p(f) = \sum_{e\in p} \bar{\ell}_e(x_e(f))$ *and*

$$C(x) = \sum_{e\in E} x_e \ell_e(x_e) \qquad \text{and} \qquad \bar{C}(x) = \sum_{e\in E} x_e \bar{\ell}_e(x_e).$$

1. *Justify that for all* $k \in [\![1, K]\!]$*, there exists* $c_k \in \mathbb{R}_+$ *such that for all path* $p \in \mathcal{P}_k$*,*
$$f_p^{UE,r} > 0 \Rightarrow \ell_p(f^{UE,r}) = c_k.$$
2. *Show that, for any* $x \in \mathbb{R}_+^{|E|}$*,* $C(x) \leq \bar{C}(x)$*, and that* $C(x^{UE,r}) = \bar{C}(x^{UE,r})$*.*
3. *Show that, for any* $x \in \mathbb{R}_+^{|E|}$*,* $x_e(\bar{\ell}_e(x_e) - \ell_e(x_e)) \leq x_e^{UE,r} \ell_e(x_e^{UE,r})$*.*
4. *Deduce that,* $\bar{C}(x^{SO,2r}) - C(x^{SO,2r}) \leq C(x^{UE,r})$*.*
5. *On the other hand, show that, for every path* $p \in \mathcal{P}_k$*,* $\bar{\ell}_p(f^{SO,2r}) \geq c_k$*.*
6. *Write* C *and* $\bar{C}$ *as function of* f *instead of* x *(we keep the same notation).*
7. *Deduce that,* $\bar{C}(f^{SO,2r}) \geq 2C(f^{UE,r})$*.*
8. *Finally, show that,* $C(f^{UE,r}) \leq C(f^{SO,2r})$*. Give an interpretation of this result.*

Exercice 4 (7pts)**.** *Consider the function* $f(x_1, x_2) = 4x_1^4 - 2x_1 + x_2^2 - x_2 + 2$*, and the set*

$$X = \left\{x \in \mathbb{R}^2_+ \quad | \quad 2x_2 + x_1 \leq 2\right\}$$

and $x^0 = (0, 0)$*. A scheme of* X *representing the iteration and search direction of the algorithm might be helpful.*

1. *Justify that* X *is polyhedral and find its extreme points.*
2. *Compute* ∇f
3. *Justify that this problem can be solved by Frank-Wolfe (aka conditional gradient) algorithm.*
4. *Find the descent direction* d^0 *of the Frank-Wolfe algorithm starting from* x_0*.*
   (hint : use the extreme points of X*).*
5. *Find the optimal step* t^0 *of the first step of Frank-Wolfe algorithm. What is the new point* x^1 *?*
6. *What is the upper and lower bound obtained along this first iteration ?*
7. *Find the descent direction* d^1 *of the second step of Frank-Wolfe algorithm.*
8. *Write the unidimensional optimisation problem that would determine the next optimal step* t^1 *(do not solve it).*
9. *Compute the lower bound associated to the second step of the algorithm.*

Exercice 5 (Bonus)**.** *According to Yuso, why is the package transport problem more complicated than the taxi problem ? Why are Yuso solving shortest path problem for ?*